Partisan Subtraction Games

Working Notes: Comments welcome!

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February 12, 1995

Contents

1 Introduction

Our basic definitions follow those of Fraenkel and Kotzig, whom we quote (almost verbatim) from the abstract to their paper [FrKo87]:

> A subtraction game $S = (s_1, \ldots, s_k)$ is a two-player game of complete information that can be played using piles of tokens. At his turn, a player removes a number m of tokens from some pile, provided $m \in S$. The player first unable to move loses. This impartial game becomes partisan if, instead of one set S, two finite sets S_L and S_R are given: Left removes tokens as specified by S_L, and Right as specified by S_R. We say that S_L *dominates* S_R (and write $S_L \succ S_R$) if for all sufficiently large piles Left wins both as first and second player.

Fraenkel and Kotzig show that the dominance relation is nontransitive on the class of sets S and give

$$(1, 2, 6) \prec (1, 3, 5) \prec (2, 3, 4) \prec (1, 2, 6)$$

as an example. They also observe that every partisan subtraction game is *periodic* in the sense that its single heap outcome classes eventually fall into a repeating pattern as the pile size increases.

Here, we study *sums* of partisan subtraction games, under certain conditions finding exact game values and atomic weights [Con76]. Some particular games are solved completely; others only partially.

2 Definitions and background

This section summarizes background combinatorial game definitions and facts with little further explanation. For more information, see Conway [Con76], or Berlekamp, Conway and Guy [BCH82].

2.1 Example games

Some numbers and games we will encounter are as follows:

0	zero	$\{\,\vert\,\}$
1	one	$\{0\,\vert\,\}$
2	two	$\{1\,\vert\,\} = \{0,1\,\vert\,\} = 1+1$
3	three	$\{2\,\vert\,\} = \{0,1,2\,\vert\,\} = 1+1+1$
$\frac{1}{2}$	one-half	$\{0\,\vert\,1\}$
$*$	star	$\{0\,\vert\,0\} = *1$
$*2$	star two	$\{0,*\,\vert\,0,*\}$
$2*$	two star	$\{2\,\vert\,2\} = 2+*$
$\uparrow$	up	$\{0\,\vert\,*\}$
$\downarrow$	down	$\{*\,\vert\,0\}$
$\uparrow*$	up star	$\{0,*\,\vert\,0\} = \uparrow + *$
$\downarrow*$	down star	$\{0\,\vert\,0,*\} = \downarrow + *$
$\Uparrow$	double up	$\{0\,\vert\,\uparrow*\} = \uparrow + \uparrow$
$\Uparrow*$	double up star	$\{0\,\vert\,\uparrow\} = \uparrow + \uparrow + *$
$\uparrow^2$	up second	$\{0\,\vert\,\downarrow*\}$

2.2 Outcome classes

The following miscellaneous facts and definitions regarding outcome classes and the simplification of games [BCG82], [Con76] will be employed throughout our work:

$G > 0$ or G is **positive**	if Left wins in best play
$G < 0$ or G is **negative**	if Right wins in best play
$G = 0$ or G is **zero**	if player 2 (i.e. whoever moves second) wins in best play
$G \parallel 0$ or G is **fuzzy**	if player 1 wins in best play.

$G \geq 0$ means Left wins, provided Right starts.
$G \leq 0$ means Right wins, provided Left starts.

$G > H$ means that G is *better* than H for Left.
$G < H$ means that G is *worse* than H for Left.

G positive or fuzzy, H positive or zero implies $G + H$ positive or fuzzy.

2.3 Reversible move simplification

If any Right option G^R of G has itself a Left option $G^{RL} \geq G$, then the value of G is unchanged by replacing G^R as a Right option of G by all the Right options $X, Y, Z, \ldots$ of that G^{RL}.

If any Left option G^L of G has itself a Right option $G^{LR} \leq G$, then the value of G is unchanged by replacing G^L as a Left option of G by all the Left options $X, Y, Z, \ldots$ of that G^{LR}.

2.4 Dominated option simplification

If $G = \{A, B, C, \ldots \,|\, D, E, F, \ldots\}$ and $A \leq B$, then A is dominated by B, and the option A may be dropped from G without affecting its value.

If $G = \{A, B, C, \ldots \,|\, D, E, F, \ldots\}$ and $D \leq E$, then E is dominated by D, and the option E may be dropped from G without affecting its value.

Note that dominated option simplification and Fraenkel and Kotzig's $\succ$ relation are different notions. In fact, we shall not be interested in the dominance relation at all for the remainder of this paper.

3 The games $S_L = (1, 2)$, $S_R = (1, k)$

3.1 The game $S_L = (1, 2)$, $S_R = (1, 3)$

Let G_n be the one pile, n token game with $S_L = (1, 2)$, and $S_R = (1, 3)$. Computation of initial game values for small n yields the following results:

n	G_n
0	$\{\,|\} = 0$
1	$\{G_0 \,|\, G_0\} = \{0 \,|\, 0\} = *$
2	$\{G_0, G_1 \,|\, G_1\} = \{0, * \,|\, *\} = \{0 \,|\, *\} = \uparrow$
3	$\{G_1, G_2 \,|\, G_0, G_2\} = \{*, \uparrow \,|\, 0, \uparrow\} = \{0, * \,|\, 0\} = \uparrow * = \uparrow + *$
4	$\{G_2, G_3 \,|\, G_1, G_3\} = \{\uparrow, \uparrow * \,|\, *, \uparrow *\} = \{\uparrow \,|\, *\} = \uparrow 2 = \uparrow + \uparrow^2$

Here, for example, in computing $G_2 = \{0, * \,|\, *\} = \{0 | *\}$ we have reversed Left's G_2-option to the game $G_1 = *$ through ${G_1}^R = 0$ because $0 \leq G_2$ (i.e., Left wins G_2 provided Right starts). In fact, G_2 is strictly positive (Left wins no matter who starts). Dominated option simplification of these expressions has performed at several points as well.

Problem 1 *Show that the game values G_n for $n \geq 5$ satisfy the simpler recursion $G_n = \{0 \,|\, G_{n-3}\}$.*

The values G_n fall into a period 3 pattern, as follows:

	0	1	2
0+	0	$*$	$\uparrow$
3+	$\uparrow + *$	$\uparrow 2$	$2 \cdot \uparrow + *$
6+	$2 \cdot \uparrow$	$2 \cdot \uparrow 2 + *$	$3 \cdot \uparrow$
9+	$3 \cdot \uparrow + *$	$3 \cdot \uparrow 2$	$4 \cdot \uparrow + *$
12+	$4 \cdot \uparrow$	$4 \cdot \uparrow 2 + *$	$5 \cdot \uparrow$
15+	$5 \cdot \uparrow + *$	$5 \cdot \uparrow 2$	$6 \cdot \uparrow + *$
18+	$6 \cdot \uparrow$	$6 \cdot \uparrow 2 + *$	$7 \cdot \uparrow$
21+	$\cdots$		

Note on reading table entries: our game notation follows that found for example in [Conw76, page 194ff]. The game

$$2 \cdot \uparrow = \uparrow + \uparrow$$

is *not* the same game as

$$\uparrow 2 = \uparrow + \uparrow^2 .$$

Multiplicative notation takes precedence over its additive counterpart so that

$$G_{13} = 4 \cdot \uparrow 2 + * = \uparrow + \uparrow^2 + \uparrow + \uparrow^2 + \uparrow + \uparrow^2 + \uparrow + \uparrow^2 + *,$$

and

$$G_9 = 4 \cdot \uparrow + * = \uparrow + \uparrow + \uparrow + \uparrow + * .$$

Example 1 *Two players, Left and Right, play a sum of partisan subtraction games as follows. Before them are several piles of colored tokens, each such pile being made up entirely of bLue tokens ($S_L = (1,2)$, $S_R = (1,3)$), or alternatively entirely of Red tokens ($S_L = (1,3)$, $S_R = (1,2)$). From the position G*

7	*3*	*4*	*1*
Red	*bLue*	*bLue*	*Red*

who has the advantage, and what are the winning moves?

Solution: From left to right, the four summands are

$$\begin{aligned} 7 \text{ Red} &= 2 \cdot \downarrow 2 + * \\ 3 \text{ bLue} &= \uparrow + * \\ 4 \text{ bLue} &= \uparrow 2 \\ 1 \text{ Red} &= * \end{aligned}$$

Forming their sum, we obtain

$$(\downarrow + \downarrow_2 + \downarrow + \downarrow_2 + *) + (\uparrow + *) + (\uparrow + \uparrow^2) + *,$$

or

$$\downarrow_2 + * .$$

Although $\downarrow_2$—the negative of $\uparrow^2$—is less than zero, the result of adding $*$ to $\downarrow_2$ is fuzzy. Therefore both Left and Right have winning moves from G, but only if each is allowed to move first. For Right, a winning move is to take the single token from the fourth Red pile, transforming G into a position of value $\downarrow_2$. Left wins from G by reducing the 7-token Red heap by one token, reaching $2 \cdot \downarrow$ in this pile so that the total value becomes

$$(\downarrow + \downarrow) + (\uparrow + *) + (\uparrow + \uparrow^2) + *,$$

or simply

$$\uparrow^2 .$$

$\blacksquare$

3.2 The game $S_L = (1,2)$, $S_R = (1,4)$

The recursion for the n-heap game value y_n is

$$y_n = \{y_{n-1}, y_{n-2} | y_{n-1}, y_{n-4}\}, n \geq 4$$

Computing initial values, we obtain

```
y0 := 0                y12:= { 0 | y8 }
y1 := { 0 | y0 }       y13:= { 0 | y9 }
y2 := { 0 | y1 }       y14:= { 0 | y10 }
y3 := { 0 | y2 }       y15:= { 0 | y11 }
y4 := {y3 | 0  }       y16:= { 0 | y12 }
y5 := { 0 | y1 }       y17:= { 0 | y13 }
y6 := { 0 | y2 }       y18:= { 0 | y14 }
y7 := { 0 | y3 }       y19:= { 0 | y15 }
y8 := { 0 | y4 }       y20:= { 0 | y16 }
y9 := {y3 | y5 }       y21:= { 0 | y17 }
y10:= { 0 | y6 }       y22:= { 0 | y18 }
y11:= { 0 | y7 }       y23:= { 0 | y19 }
```

The first game we haven't seen before is y_4, which is

$$G = \{\Uparrow * | 0\}.$$

We can give simple closed forms for all the game values by introducing the game $\uparrow^{2+} = \{\uparrow \mid \downarrow *\}$, which is pronounced "up-second-plus." [Con76]. The game values for $S_L = (1,2)$, $S_R = (1,4)$ break up into periods of length 4:

	0	1	2	3
0+	0	$*$	$\uparrow$	$2 \cdot \uparrow + *$
4+	$1.(\uparrow * + \uparrow^{2+})$	$\uparrow$	$2 \cdot \uparrow + *$	$3 \cdot \uparrow$
8+	$2.(\uparrow * + \uparrow^{2+})$	$\uparrow + 1.(\uparrow 2 + *)$	$3 \cdot \uparrow$	$4 \cdot \uparrow + *$
12+	$3.(\uparrow * + \uparrow^{2+})$	$\uparrow + 2.(\uparrow 2 + *)$	$4 \cdot \uparrow + *$	$5 \cdot \uparrow$
16+	$4.(\uparrow * + \uparrow^{2+})$	$\uparrow + 3.(\uparrow 2 + *)$	$5 \cdot \uparrow$	$6 \cdot \uparrow + *$

3.3 Atomic weights and game values for $k \geq 5$

For values $k \geq 5$ the situation is more complex. The following tables summarize the results of atomic weight computations for $3 \leq k \leq 9$.

Atomic weight

$k = 3$

	0	1	2
0+	0	0	1
3+			

$k = 4$

	0	1	2	3
0+	0	0	1	2
4+				

$k = 5$

	0	1	2	3	4
0+	0	0	1	2	3
5+	3/2	3/2			
10+	3	3			
15+					

$k = 6$

	0	1	2	3	4	5
0+	0	0	1	2	3	4
6+	$2*$	$2*$				
12+	4	4				
18+						

$k = 7$

	0	1	2	3	4	5	6
0+	0	0	1	2	3	4	5
7+	$\{3\|2\}$	$\{3\|2\}$	2				
14+	5	5	7/2				
21+			5				

$k = 8$

	0	1	2	3	4	5	6	7
0+	0	0	1	2	3	4	5	6
8+	$\{4\|2\}$	$\{4\|2\}$	2	3				
16+	6	6	$4*$	9/2				
24+			6	6				

$k = 9$

	0	1	2	3	4	5	6	7	8
0+	0	0	1	2	3	4	5	6	7
9+	$\{5\|2\}$	$\{5\|2\}$	2	3					
18+	7	7	$\{5\|4\}$	$5*$					
27+			7	7					

To extend an atomic weight table down a column, add one at each row to the last (always integral) atomic weight given in that column. For example, for $k = 6$ the atomic weight of a heap of 16 (= 12+4) counters is 5 (= 3 at top of column, plus 2 rows down).

3.3.1 Game values

Let $y = y_{n,k}$ be the game value of the n token heap in $S_L = (1, 2)$, $S_R = (1, k)$, and write $y^- = y_{n-k,k}$ when $n \geq k$, and $y^- = 0$ otherwise. Fixing k, these game values appear to eventually fall into the recursion

$$y = \{0|y^-\}$$

for all sufficiently large n, with

$$y > \{0|y^-\}$$

being the case whenever the previous equality fails.

Let $\Omega(k)$ stand for the last exceptional case (for which $y > \{0|y^-\}$). We obtained the following sequence:

k	3	4	5	6	7	8	9	10	11	12	13	14	15
$\Omega(k)$	4	9	12	15	24	28	32	45	50	55	72	78	84

In fact, $\Omega(k)$ is given exactly by the following quadratic equations:

$$\begin{aligned} \Omega(3l) &= 9l^2 - 16l + 11 \\ \Omega(3l+1) &= 9l^2 - 12l + 12 \\ \Omega(3l+2) &= 9l^2 - 11l + 14 \end{aligned}$$

Often even more is true, with one of the equations

$$y = \{0|y^-\} = y^- + \uparrow *$$

or

$$y = \{0|y^-\} = y^- + \uparrow 2*,$$

being valid. The following game difference tables give more specific information for small values of k.

Game values $y - y^{-}$

$k = 3$

	0	1	2
0+	0	$*$	$\uparrow$
3+	$\uparrow *_{>}$	$\uparrow 2 *_{>}$	$\uparrow *_{=}$
6+	$_{=}$	$\cdots$	

$k = 4$

	0	1	2	3
0+	0	$*$	$\uparrow$	$\Uparrow *$
4+	$\uparrow 2 * \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
8+	$_{=}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$

$k = 5$

	0	1	2	3	4
0+	0	$*$	$\uparrow$	$\Uparrow *$	$3\uparrow$
5+	$\Uparrow *, 3\uparrow \vert 0_{>}$	$3\uparrow * \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
10+	$\Uparrow *, 3\uparrow \vert 0_{=}$	$3\uparrow * \vert 0_{>}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$
15+	$\dagger$	$\dagger$			
20+					

$k = 6$

	0	1	2	3	4	5
0+	0	$*$	$\uparrow$	$\Uparrow *$	$3\uparrow$	$4\uparrow *$
6+	$3\uparrow, 4\uparrow * \vert 0_{>}$	$4\uparrow \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
12+	$3\uparrow, 4\uparrow * \vert 0_{=}$	$4\uparrow, 4\uparrow \vert 0 \Vert 0_{>}$	$\uparrow 2 *_{>}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$
18+	$\dagger$	$\dagger$				

$k = 7$

	0	1	2	3	4	5	6
0+	0	$*$	$\uparrow$	$\Uparrow *$	$3\uparrow$	$4\uparrow *$	$5\uparrow$
7+	$4\uparrow *, 5\uparrow \vert 0_{>}$	$5\uparrow * \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
14+	$4\uparrow *, 5\uparrow \vert 0_{=}$	$5\uparrow *, 5\uparrow * \vert 0 \Vert 0_{>}$	$3\uparrow * \vert 0_{>}$	$\uparrow *_{=}$	$_{=}$	$_{=}$	$_{=}$
21+	$\dagger_{=}$	$\dagger_{=}$	$3\uparrow * \vert 0_{=}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$	
28+	$\dagger$	$\dagger$	$\dagger$				

$k = 8$

	0	1	2	3	4	5	6	7
0+	0	$*$	$\uparrow$	$\Uparrow *$	$3\uparrow$	$4\uparrow *$	$5\uparrow$	$6\uparrow *$
8+	$5\uparrow, 6\uparrow * \vert 0_{>}$	$6\uparrow \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
16+	$5\uparrow, 6\uparrow * \vert 0_{=}$	$6\uparrow, 6\uparrow \vert 0 \Vert 0_{>}$	$3\uparrow *, 4\uparrow \vert 0_{>}$	$3\uparrow * \vert 0_{>}$	$\uparrow *_{=}$	$_{=}$	$_{=}$	$_{=}$
24+	$\dagger_{=}$	$\dagger_{=}$	$3\uparrow *, 4\uparrow \vert 0_{=}$	$3\uparrow * \vert 0_{=}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$	
32+	$\dagger$	$\dagger$	$\dagger$	$\dagger$				

$k = 9$

	0	1	2	3	4	5	6	7	8
0+	0	$*$	$\uparrow$	$\Uparrow *$	$3\uparrow$	$4\uparrow *$	$5\uparrow$	$6\uparrow *$	$7\uparrow$
9+	$6\uparrow *, 7\uparrow \vert 0_{>}$	$7\uparrow * \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$\uparrow *_{=}$
18+	$6\uparrow *, 7\uparrow \vert 0_{=}$	$7\uparrow *, 7\uparrow * \vert 0 \Vert 0_{>}$	$4\uparrow, 5\uparrow * \vert 0_{>}$	$4\uparrow \vert 0_{>}$	$\uparrow *_{=}$	$\uparrow *_{=}$	$_{=}$	$_{=}$	$_{=}$
27+	$\dagger_{=}$	$\dagger_{=}$	$4\uparrow, 5\uparrow * \vert 0_{=}$	$4\uparrow, 4\uparrow \vert 0 \Vert 0_{>}$	$\uparrow 2 *_{>}$	$\uparrow 2 *_{>}$	$_{=}$	$\cdots$	
36+	$\dagger$	$\dagger$	$\dagger$	$\dagger$					

The "$>$" or "$=$" subscripts in the tables indicate whether $y > \{0|y^{-}\}$ or $y = \{0|y^{-}\}$ is the case at a particular entry.

To extend a game value difference table down a column that doesn't end with a dagger symbol (†), simply repeat the last given value in that column (always $\uparrow 2*$ or $\uparrow *$). Since the table entries are differences, computing the actual game value for a given heap size involves adding to the given difference table entry the entries in all rows above the desired entry in the same column. For example, the value of a heap of 31 tokens in $(1, 2)$ vs. $(1, 7)$ is

$$\uparrow 2* + \Uparrow * + \uparrow * + \uparrow * + \uparrow 2* = 6\uparrow + 2\uparrow^2 + * .$$

3.3.2 Extending the daggers

Columns that end in daggers may be extended by use of the equation

$$y = \{0|y^-\},$$

and their atomic weights increase by one at each row; however, both the resulting canonical game expressions and associated difference table entries appear to be complicated objects.

Can these values be systematized in some way? Here is one observation. Fix $k = 5$ and consider the leftmost column in its table, beginning at heap size 10 (and write y_{10}, y_{15}, y_{20}, etc) for these games. Then

$$\begin{aligned} y_{10} + 3\downarrow &= 0||0, \downarrow *|3\downarrow \\ y_{15} + 4\downarrow * &= 0|||0, \downarrow *||\downarrow *, \Downarrow|4\downarrow \\ y_{20} + 5\downarrow &= 0||||0, \downarrow *|||\downarrow *, \Downarrow||\Downarrow, 3\downarrow *|5\downarrow \end{aligned}$$

A similar equation seems to apply to the 1 mod 5 entries in the $k = 5$ table. Does this continue?

4 References

FrKo87 A. S. Fraenkel and A. Kotzig, "Partizan Octal Games: Partizan Subtraction Games," in *International Journal of Game Theory,* **16**, Issue 2, pages 145-154.

BCG82 E. R. Berlekamp, J. H. Conway, and R. K. Guy, Winning Ways for your Mathematical Plays. Academic Press, London.

Con76 J. H. Conway, On Numbers and Games. Academic Press, London.