How to lose at **.123**

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The octal game **.123** can be played with counters arranged in heaps. Two players take turns removing one, two or three counters from a heap, subject to the following additional conditions:

1. Three counters may be removed from any heap;
2. Two counters may be removed from a heap, but only if it has more than two counters; and
3. One counter may be removed only if it is the only counter in that heap.

In *normal play* of .123, the last player able to make a legal move is declared the winner. In normal play, each heap size reduces to a nim-heap. The resulting nim sequence[1] is periodic of length 5, beginning at heap 5.

+	1	2	3	4	5
0+	1	0	2	2	1
5+	0	0	2	1	1
10+	0	0	2	1	1
15+	···				

Table 1
Normal play nim values of .123

In *misère play*, the last player to make a legal move is declared to be the *loser* of the game.

[1] See *Winning Ways* (henceforth simply WW), Chapter 4, pg 97, "Other Take-Away Games;" also Table 7(b), pg 104.

Taking our notation again from *Winning Ways* (now Chapter 13, "Survival in the Lost World"), we exhibit the genus sequence of .123 in Table 2. This sequence is also periodic of length 5:

+	1	2	3	4	5
0+	1	0	2	2	1
5+	0^0	0	2^{1420}	1^{20}	1
10+	0^0	0	2^{1420}	1^{20}	1
15+	$\cdots$				

Table 2
G*-values of .123

In Table 2, a entry that is a simple integer (0, 1, or 2) represents a nim heap of the corresponding size. Nim heaps (and sums of nim heaps) are always *tame* games in misère play. The genus symbols[2] of the nim heaps that occur in Table 2 are

$$0 = 0^{1202020\cdots} \quad (1)$$
$$1 = 1^{0313131\cdots} \quad (2)$$
$$2 = 2^{2020202\cdots} \quad (3)$$

In misère play of .123, the first non-nim-heap occurs at the six-counter heap. It is the game $a = 2_+ = \{2\}$. The eight-counter heap is $b = \{a, 1\}$, and the nine-counter heap is $c = \{a, 0\}$. Although the subsequent games occurring at heap sizes = 1, 3, and 4 (modulo 5) are not identical to a, b, and c, respectively, their respective genera do repeat, as indicated in Table 2.

Here's what we can (and cannot) do with Table 2:

Single heaps

We *can* determine the outcome class of *single-heap* .123 positions. The first superscript in a heap's genus symbol is 0 if and only iff that heap size is a P-position. The single heap P-positions of .123 therefore occur at heap sizes

$$1, 5, 6, 10, 11, 15, 16, 20, 21, \ldots$$

For example, the heap of size 7 has its first superscript = 1. It is therefore an N-position. The winning move is $7 \to 5$.

Multiple heaps

We *cannot* immediately determine the outcome class of *multiple-heap* .123 positions using Table 2. However, Table 2 does provide a basis for investigating multiheap positions. For example, here is a table that shows the genera of two-heap positions up to heap size nine:

[2] WW pg 402, top.

+	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
h_1	0	1	3	3	0	1^{13}	1	3^{0531}	0^{31}
h_2		0	2	2	1	0^0	0	2^{1420}	1^{20}
h_3			0^0	0^0	3	2^{20}	2	0^{420}	3^{02}
h_4				0^0	3	2^{20}	2	0^{420}	3^{02}
h_5					0	1^{13}	1	3^{0531}	0^{31}
h_6						0^0	0^0	2^{20}	1^{13}
h_7							0	2^{1420}	1^{20}
h_8								0^{12}	3^{02}
h_9									0^{02}

Table 3
Some G*-values of games $h_i + h_j$

Here's how the genus of a particular sum $G = h_8 + h_5$ was computed from earlier values in Table 3. First, we rewrite genus(G) in terms of its options:

$$\text{genus}(G) = \text{genus}(h_8 + h_5) = \text{genus}(\{h_6 + h_5, h_5 + h_5, h_8 + h_3, h_8 + h_2\})$$

The genus of a non-empty game $G = \{A, B, \cdots\}$ can be calculated from the genus of its options $A, B, \ldots$ using the *mex-with-*$(\gamma, \gamma+1)$*-carrying* algorithm[3] (x symbols represent no carry):

$$\begin{aligned}
\text{carry}_{\gamma} &= x^{x04202} \\
\text{carry}_{\overset{*}{\gamma+1}} &= x^{x15313} \\
\text{genus}(h_6 + h_5) &= 1^{131313\ldots} \\
\text{genus}(h_5 + h_5) &= 0^{120202\ldots} \\
\text{genus}(h_8 + h_3) &= 0^{420202\ldots} \\
\text{genus}(h_8 + h_2) &= \underline{2^{142020\ldots}} \\
\text{genus}(G) &= 3^{053131\ldots}
\end{aligned}$$

Tame, Restive, Restless or Wild?

Another thing we can do with Table 2 is classify the individual heap sizes as *tame*, *restive*, *restless*, or *wild*:

1. All the positions equivalent to nim heaps (ie, $h_1 = 1$, $h_2 = 0$, $h_3 = 2$, $h_4 = 2$, $h_{5k} = 1$, and $h_{5k+2} = 0$) are *tame*.

2. The game $h_6 = 2_+$ has genus 0^0 and a single tame option $h_4 = h_3 = 2$. It is therefore[4] also tame. The positions h_{5k+1} for $k \geq 1$ have genus 0^0 too, and also happen to be tame, but this is not because they have

[3] See the more complete description of this algorithm in the section titled *"But What if They're Wild?" asks the Bad Child* in WW, (page 410).
[4] WW pg 405 top.

tame options (they don't). To show h_{5k+1} is tame, we instead exhibit appropriate *reverting moves*[5] from h_{5k+1}'s two possible initial moves. The first of these moves is

$$h_{5k+1} \to h_{5k+1-2}$$

to a game of genus 1^{20} (with reverting moves to games of genus 0^1 and 0^0), and the second is

$$h_{5k+1} \to h_{5k+1-3}$$

to a game of genus 2^{1420} (with reverting moves to games of genus 0^0 and 1^0).

3. Passing over h_8 for the moment, we turn next to h_9, of genus 1^{20}. This is a *restive* game[6]. Why? First, it has the correct the genus—for a game G to be restive, its nim values g^γ have to be one of the *restive pairs* in which g is 0 or 1 and γ is 2 or more:

$$0^0, 0^3, 0^4, 0^5, \ldots \text{ or } 1^2, 1^3, 1^4, 1^5, \ldots$$

There is no further condition if all the options of G are tame[7]. Since these conditions are met by h_{5k+4} for every $k \geq 1$, all these games are restive. What can we conclude from this information about h_{5k+4}?

   (a) **Sums of a single h_{5k+4} with nim heaps in .123 are tame**. This is an application of the Intermediate Value Theorem[8].

   (b) **We can easily obtain the genus of such a sum**. If n is a nim heap chosen from $\{0,1,2,3\}$, the genus of $h_{5k+4} + n$ is $(n \overset{*}{+} 1)^{n \overset{*}{+} 2}$.

   (c) **All even multiples of a heap h_{5k+8} have genus** $0^{1202\cdots}$. This is a consequence of the Noah's Ark Theorem.

4. The genus of h_8 is 2^{1420}, and it is *restless*.[9] In fact, the same is true of all the games h_{5k+8} for $k \leq 1$.

Since h_6 is a tame game of the same genus 0^{02} as the "adder" game $: 4 = h_3 + h_3$, it's tempting to think that each h_6 can be treated exactly as if it were instead two nim heaps of size two (see the following section for more information on adders).

This would be a mistake, however. The sum $s_1 = h_9 + h_6$ has genus 1^{13}, but $s_2 = h_9 + h_3 + h_3$ has genus 1^{20}. In particular, adding a nim heap of size 1 to s_1 gives a misère P-position, while doing the same to s_2 yields an N-position. This example shows the care that must be taken before drawing any conclusions about sums involving a non-nim-heap tame position such as h_6 and a restive game such as h_9.

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[5]WW pg 405 top, but the next paragraph.
[6]WW pg 405, bottom.
[7]WW pg 406.
[8]WW pg 406, bottom.
[9]WW pg 405, bottom.

Adders

The *adder* games $:a$ for $a \geq 0$ are defined by setting $:0 = 0$, $:1 = 1$, and $:2 = 2$ (the misère nim heaps of size 0, 1, and 2, respectively). In general

$$:(2a) = \underbrace{:2 + :2 + \cdots + :2}_{a}$$

and

$$:(2a+1) = :(2a) + :1$$

In terms of options,

$$\begin{array}{rcl}
:0 &=& 0 \\
:1 &=& 1 \\
:2 &=& 2 \\
:3 &=& 3 \\
:4 &=& \{:2, :3\} \\
:5 &=& \{:2, :3, :4\} \\
:6 &=& \{:4, :5\} \\
:7 &=& \{:4, :5, :6\} \\
:8 &=& \{:6, :7\} \\
:9 &=& \{:6, :7, :8\} \\
:10 &=& \{:8, :9\} \\
:11 &=& \{:8, :9, :10\} \\
\cdots & & \cdots
\end{array}$$

In terms of addition,

$$\begin{array}{rcl}
:a + :b &=& :(a+b) \text{ if either } a \text{ or } b \text{ (or both) is even} \\
:a + :b &=& :(a+b-2) \text{ if } a \text{ and } b \text{ are both odd.}
\end{array}$$

In terms of genera:

$$\begin{array}{rcl}
\text{genus}(:0) &=& 0^{12020\cdots} \\
\text{genus}(:1) &=& 1^{03131\cdots} \\
\text{genus}(:2) &=& 2^{2020\cdots} \\
\text{genus}(:3) &=& 3^{3131\cdots} \\
\text{genus}(:4) &=& 0^{0202\cdots} \\
\text{genus}(:5) &=& 1^{1313\cdots} \\
\text{genus}(:6) &=& 2^{2020\cdots} \\
\text{genus}(:7) &=& 3^{3131\cdots} \\
\cdots & & \cdots
\end{array}$$

where the last four entries repeat with period 4.

Solution to .123

To determine .123 outcome classes, pretend that each heap size is either an adder or special game C, A, or B according to Table S1. Its last five values repeat indefinitely.

	1	2	3	4	5
0+	:1	:0	:2	:2	:1
5+	C	:0	A	B	:1
10+	C	:0	A	B	:1
15+	...				

Table S1
Heap equivalences for .123

The genera of individual C, A, and B games are given in Table S2:

C	A	B
0^{02}	2^{1420}	1^{20}

Table S2
Genera for heaps of the form
$C = 5k + 1$, $A = 5k + 3$, and $B = 5k + 4$, for $k \geq 1$

If a position is a sum of at most one C, A, or B and an adder :a, Tables S1 and S2 can be used to determine whether it is an N- or P-position. For a general .123 position that contains multiple A's, B's, and/or C's, reduce it to a form where Tables S1 and S2 can be applied using the following additional logic:

First replace each heap by the equivalent game from Table S1, just as before. Then, depending upon how many B's and C's result, apply exactly one of the two following reduction rules S3 and S4:

- *If there is at least one C or at least two or more B's*, ignore the C terms and reduce the A's and B's to an adder :a according to Table S3:

	even # B's	odd # B's
even # A's	:4	:5
odd # A's	:2	:3

Reduction rule S3

- *Alternatively, if there is no C and at most one B*, reduce the position to at most one A or B and/or adder according Table S4:

	no B's	one B
even # A's	:0	B
odd # A's	A	B + :2

Reduction rule S4

Example 1 *Suppose a .123 position G has heaps of size 9, 8, 5, and 3. Is the position a misère P- or N-position, and if the latter, what are the winning move(s)?*

Solution: There is no C, one B, and one A present in G. Applying reduction rule $S4$, we obtain an equivalent

$$B + :2$$

for the heaps 9 and 8 present in G. The remaining heaps of size 5 and 3 in G are equivalent to :1 and :2, respectively. The whole sum is therefore equivalent to

$$(B + :2) + (:1 + :2) = B + :5,$$

whose misère nim value can be read off as 3 from the (infinite) genus symbol $1^{202020\dots}$ for B (the third superscript of this symbol—ie, 2—nim-summed with :1 is 3). Since the misère nim value of G is non-zero, the given position is an N-position. To find a winning move, we must perform similar computations on the eight possible moves from $\{9, 8, 5, 3\}$, looking for moves that result in positions with misère nim value 0.

option	type	example rules applied	reduced form	misère nim value
{**7**,8,5,3}	{7,A,5,3}	S1/S2	:0 + A + :1 + :2 = A + :3	5
{**6**,8,5,3}	{C,A,5,3}	S3	:2 + :1 + :2 = :5	1
{9,**6**,5,3}	{B,C,5,3}	S3	:4 + :1 + :2 = :6	2
{9,**5**,5,3}	{B,5,5,3}	S1/S2	B + :1 + :1 + :2 = B + :2	0
{9,8,**3**,3}	{B,A,3,3}	S4	B + :2 + :2 + :2 = B + :6	0
{9,8,**2**,3}	{B,A,2,3}	S4	B + :2 + :0 + :2 = B + :4	2
{9,8,5,**1**}	{B,A,5,1}	S4	B + :2 + :1 + :1 = B + :2	0
{9,8,5,**0**}	{B,A,5,0}	S4	B + :2 + :1 + :0 = B + :3	1

There are three different winning moves from G. They are $8 \to 5$, $5 \to 3$, and $3 \to 1$.