Taming the Wild in Misere Impartial Games

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[1] http://arxiv.org/abs/math.CO/0501315

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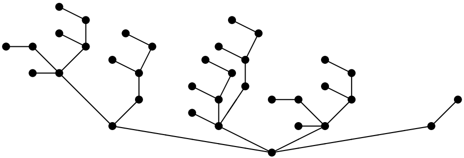


Why isn't there a misere Sprague-Grundy theory?

	Normal play	Misere play
Endgame	P-position	N-position
P-position	Additive identity	No
Inverses	$G + G = 0$	No
Canonical forms	Nim heaps $*k$	Complicated trees
Sums	Nim addition $\oplus$	Recursive mess
Options	Mex rule	Usu. unsimplifiable
Algebra	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$	Genus theory: tame/wild
Literature	Many games analyzed	Few papers
	Sprague-Grundy Theory	???

How to generalize Sprague-Grundy to misere play

Γ = Impartial game (fixed rules)

$\Downarrow$

Indistinguishability congruence ρ

$\Downarrow$

Positions modulo ρ

Normal play	Misere play
Sprague-Grundy Theory	Misere Quotient
$\mathcal{Q}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$	$\mathcal{Q}(\Gamma) = \langle$commutative semigp$\rangle$

Example

Rules of 0.123

Take: 3 beans from any heap; 2 if more than two in heap; or 1 if only one in heap. Last player *wins*.

Solution

Normal-play nim sequence:

+	1	2	3	4	5
0+	$*1$	$*0$	$*2$	$*2$	$*1$
5+	$*0$	$*0$	$*2$	$*1$	$*1$
10+	$*0$	$*0$	$*2$	$*1$	$*1$
15+	$\cdots$				

$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

Misere Play

Rules of 0.123

Take: 3 beans from any heap; 2 if more than two in heap; or 1 if only one in heap. Last player *loses*.

Solution

	1	2	3	4	5
0+	x	e	z	z	x
5+	b^2	e	a	b	x
10+	b^2	e	a	b	x
15+	$\cdots$				

Twenty-element commutative semigroup with identity e, generators $\{x, z, a, b\}$, P-positions $\{x, xa, b^2, z^2, zb\}$, and relations:

$$x^2 = a^2 = e,\ z^4 = z^2,\ b^4 = b^2,\ abz = b,\ b^3x = b^2,\ z^3a = z^2$$

Computing a misere outcome

Problem

Find the outcome class of 1+3+4+8+9+21 in misere **0.123**

Solution

$$\langle\, x^2 = a^2 = e,\ z^4 = z^2,\ b^4 = b^2,\ abz = b,\ b^3x = b^2,\ z^3a = z^2 \,\rangle$$

1	3	4	8	9	21
x	z	z	a	b	b^2

$$\begin{aligned} &= (abz)xzb^2 \quad \text{(regroup)} \\ &= (b)xzb^2 \quad \text{(relation } abz = b\text{)} \\ &= zb^2 \quad \text{(relation } b^3x = b^2\text{)} \end{aligned}$$

Since zb^2 isnt a P-position type $\{x, xa, b^2, z^2, zb\}$, it's an N-position.

Knuth-Bendix rewriting process

Input to Knuth-Bendix

A finitely presented c.s. semigroup S:

$$\langle x, z, a, b | x^2 = a^2 = e, z^4 = z^2, b^4 = b^2, abz = b, b^3x = b^2, z^3a = z^2 \rangle$$

Output from Knuth-Bendix

A *confluent rewriting system* for S:

$$[[x^2, e], [a^2, e], [z^4, z^2], [ab, zb], [z^2b, b], [b^3, xb^2], [z^2a, z^3]]$$

http://www.gap-system.org/

Using the confluent rewriting system

$$[[x^2, e], [a^2, e], [z^4, z^2], [ab, zb], [z^2b, b], [b^3, xb^2], [z^2a, z^3]]$$

Rewriting a monomial

$$\begin{aligned} xz^2ab^3 &= xz^2ab^3 & \\ &= xz^3b^3 & [z^2a, z^3] \\ &= x^2z^3b^2 & [b^3, xb^2] \\ &= z^3b^2 & [x^2, e] \\ &= zb^2 & [z^2b, b] \end{aligned}$$

Indistinguishability quotients in taking and breaking

Γ = taking and breaking game with *fixed rules and play convention* (normal or misere)

The heap alphabet and free semigroup of all positions

- Fix a *heap alphabet* $H = \{h_1, h_2, h_3, \cdots\}$

$$h_i = \text{ heap of size } i$$

- Write *sums* multiplicatively:

$$h_7{}^2 h_2 h_3$$

- $\mathcal{F}_H$ = free commutative semigroup of all such monomials
  = (1-1) correspondence with all positions in play of Γ
- Endgame = identity = 1

The indistinguishability relation ρ

Still fixing Γ (ie, rules + play convention)...

Quotient construction

- Let $u, v \in \mathcal{F}_H$ be positions in Γ.
- Say *u is indistinguishable from v* (and write $u \ \rho \ v$) if

$$\forall w \in \mathcal{F}_H, \ uw \text{ and } vw \text{ have same outcome}$$

- Then ρ is a *congruence* on $\mathcal{F}_H$.
- So there's a commutative *quotient semigroup*

$$\mathcal{Q}(\Gamma) = \mathcal{F}_H / \rho$$

Example: Normal play indistinguishability quotients

Question

- G, K normal play games—when indistinguishable?

Answer

- Suppose $G = *g$, and $K = *k$.
- If $g \neq k$, can distinguish between them (add G to both)
- If $g = k$, positions G and K are indistinguishable

The Γ = normal play Nim indistinguishability quotient

1	h_1	h_2	$\mathcal{F}_H$
$h_1 h_2 h_3$	$h_2 h_3 {h_6}^2$	$h_1 {h_3}^3$	
${h_3}^2$	$h_1 {h_3}^2$	$h_2 {h_3}^2$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	
$*0$	$*1$	$*2$	

Figure: Elements of $\mathcal{Q}(\Gamma) \equiv$ nim heaps $*k$ when Γ = normal Nim.

$$\text{nim heaps} = \underbrace{\{*0, *1, *2, *3, *4, *5, *6, *7, \ldots\}}_{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots}$$

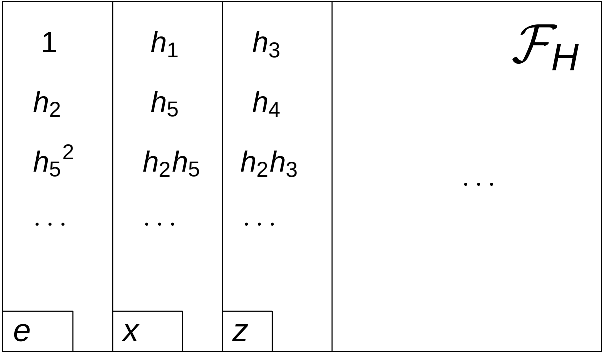
The Γ = Misere Nim indistinguishability quotient

1	h_1	${h_2}^2$	$\mathcal{F}_H$
${h_1}^2$	${h_1}^3$	$h_1 h_2 h_3$	
${h_1}^4$	${h_1}^5$	${h_2}^2{h_3}^2$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	
0^{120}	1^{031}	0^{02}	

Figure: Elements of $\mathcal{Q}(\Gamma) \equiv$ tame genera when Γ = misere Nim.

$$\text{tame genera} = \{\underbrace{0^{120}, 1^{031}}_{\mathbb{Z}_2}, \underbrace{0^{02}, 1^{13}, 2^{20}, 3^{31}, 4^{46}, 5^{57}, \ldots}_{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots}\}$$

General misere game, eg **0.123**



Three congruence classes of the (order 20) misere **0.123** quotient

congruence classes $\equiv$ elements of a commutative semigroup

Normal vs misere concepts

Normal play	Misere play
Nim heap equivalent	Quotient element
Nim addition	Quotient multiplication
N- or P-position	Quotient partition $Q(\Gamma) = P \cup N$
Nim sequence	Pretending function $\Phi : H \rightarrow Q(\Gamma)$
Mex rule	The misere mex mystery

Normal vs misere play concepts in taking and breaking games

Pretending functions & the misere mex mystery

Example: When is $4 + 4$ indistinguishable from 6 in **0.123**?

Normal **0.123**

n	1	2	3	4	5	6	7	8	9	10
$G(n)$	$*1$	$*0$	$*2$	$*2$	$*1$	$*0$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Misere **0.123** to heap 6: $\langle x, z \mid x^2 = e,\ z^3 = z \rangle$, order 6

n	1	2	3	4	5	6
$\Phi(n)$	x	e	z	z	x	z^2

Complete misere **0.123** quotient, order 20

n	1	2	3	4	5	6	7	8	9	10
$\Phi(n)$	x	e	z	z	x	b^2	e	a	b	$\cdots$

8+9 distinguishes between 4+4 and 6 in **0.123**

$$[[x^2, e], [a^2, e], [z^4, z^2], [ab, zb], [z^2 b, b], [b^3, xb^2], [z^2 a, z^3]]$$

$$\begin{aligned} (4+4)+(8+9) &= z^2(ab) \\ &= ba \\ &= zb \qquad \text{P position} \end{aligned}$$

$$\begin{aligned} (6)+(8+9) &= b^2(ab) \\ &= b^3 a \\ &= xb^2 a \\ &= (xb)(ba) \\ &= (xb)(zb) \\ &= xzb^2 \qquad \text{N position} \end{aligned}$$

Fortune cookies & the algebra of misere play

Pick your favorite taking and breaking game Γ.

$$\mathcal{Q}(\Gamma) = \mathcal{F}_H/\rho$$

Open your fortune cookie.

Your misere $\mathcal{Q}(\Gamma)$ is *finite*.

Your pretending functions have *equal period lengths*.

Your misere $\mathcal{Q}(\Gamma)$ includes your normal $\mathcal{Q}(\Gamma)$ as a *subgroup*.

#	element	genus	outcome	x	z	a	b
1	e	0^{120}	N	2	3	4	5
2	x	1^{031}	P	1	6	7	8
3	z	2^{20}	N	6	9	10	11
4	a	2^{1420}	N	7	10	1	11
5	b	1^{20}	N	8	11	11	12
6	xz	3^{31}	N	3	13	14	15
7	xa	3^{0531}	P	4	14	2	15
8	xb	0^{31}	N	5	15	15	16
9	z^2	0^{02}	P	13	17	17	5
10	za	0^{420}	N	14	17	3	5
11	zb	3^{02}	P	15	5	5	18
12	b^2	0^{02}	P	16	18	18	16
13	xz^2	1^{13}	N	9	19	19	8
14	xza	1^{531}	N	10	19	6	8
15	xzb	2^{13}	N	11	8	8	20
16	xb^2	1^{13}	N	12	20	20	12
17	z^3	2^{20}	N	19	9	9	11
18	zb^2	2^{20}	N	20	12	12	20
19	xz^3	3^{31}	N	17	13	13	15
20	xzb^2	3^{31}	N	18	16	16	18
			rank	20	16	20	12

The twenty congruence classes of misere **0.123**.

$\mathcal{Q}_{0.123}$ as a semigroup of transformations

$1 \rightleftarrows 2 \quad 3 \rightleftarrows 6 \quad 4 \rightleftarrows 7 \quad 5 \rightleftarrows 8 \quad 9 \rightleftarrows 13$

$10 \rightleftarrows 14 \quad 11 \rightleftarrows 15 \quad 12 \rightleftarrows 16 \quad 17 \rightleftarrows 19 \quad 18 \rightleftarrows 20$

The action of x on $\mathcal{Q}_{0.123}$

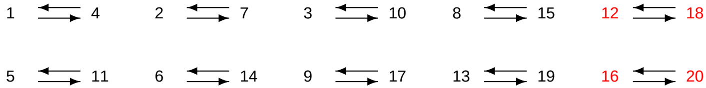
$1 \rightarrow 3 \rightarrow 9 \rightleftarrows 17 \leftarrow 10 \leftarrow 4$

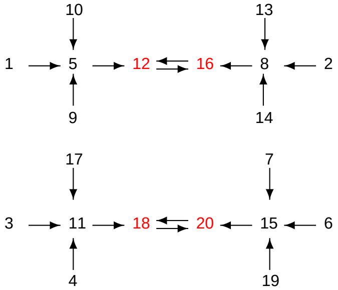
$2 \rightarrow 6 \rightarrow 13 \rightleftarrows 19 \leftarrow 14 \leftarrow 7$

$5 \rightleftarrows 11 \quad 8 \rightleftarrows 15 \quad 16 \rightleftarrows 20 \quad 12 \rightleftarrows 18$

The action of z on $\mathcal{Q}_{0.123}$

Transformation semigroup (cont'd)





The action of a (top) and b (bottom) on $\mathcal{Q}_{\mathbf{0.123}}$

Ideals

An *ideal* I of a c.s. S satisfies $IS \subseteq I$.
Let $I = \{12, 16, 18, 20\} = \{b^2, xb^2, zb^2, xzb^2\}$.

- The subset I is the *minimal ideal* of $Q_{\mathbf{0.123}}$.
- Elements of I have minimal rank = 4
- In general, for a finite c.s. S:
  - S always has a minimal ideal, which is non-empty.
  - The minimal ideal = elements with smallest *rank*
    - $rank(u) = |uS|$.
  - The minimal ideal is a *group*.

In $\mathcal{Q}_{\mathbf{0.123}}$,

$$I = \mathbb{Z}_2 \times \mathbb{Z}_2,$$ the normal play quotient

Principal ideal series

A chain of ideals with no ideal between S_i and S_{i+1}.

$$\mathcal{Q}_{\mathbf{0.123}} = S = S_1 \supset S_2 \supset S_3 \supset S_4 \supset S_5 \supset S_6 = \emptyset,$$

$$\begin{aligned}
S_1 &= S_2 \cup \{e, x, a, xa\}. \\
S_2 &= S_3 \cup \{z, xz, za, xza\} \\
S_3 &= S_4 \cup \{z^2, xz^2, z^3, xz^3\} \\
S_4 &= S_5 \cup \{b, xb, zb, xzb\} \\
S_5 &= \{b^2, xb^2, zb^2, xzb^2\} \\
S_6 &= \emptyset.
\end{aligned}$$

Partial ordering of idempotents & maximal subgroups

Idempotent: An element f satisfying $f^2 = f$.
Natural partial ordering of idempotents: $g \leq f$ iff $fg = g$.
Mutual divisibility: $u \; \eta \; v$ iff $u|v$ and $v|u$.

$$
\begin{aligned}
S_1 &= S_2 \cup \{\boxed{e}, x, a, xa\}. \\
S_2 &= S_3 \cup \{z, xz, za, xza\} \\
S_3 &= S_4 \cup \{\boxed{z^2}, xz^2, z^3, xz^3\} \\
S_4 &= S_5 \cup \{b, xb, zb, xzb\} \\
S_5 &= \{\boxed{b^2}, xb^2, zb^2, xzb^2\} \\
S_6 &= \emptyset.
\end{aligned}
$$

In a commutative semigroup, mutual divisibility is a congruence. The congruence class containing an idempotent f is precisely the maximal subgroup of S for which that element is the identity.

Rees congruence

Let I be an ideal of a commutative semigroup S.

- The *Rees congruence modulo I* is a relation η on S:

$$r \ \eta \ s \quad \text{iff} \quad \begin{cases} r = s, \text{ or} \\ \text{both } r, s \in I. \end{cases}$$

- Then
  - η is a congruence on S.
  - I collapses to a single (zero) element in S/η.
  - Elements outside I retain their identity.
- The quotient S/η is called a *Rees factor*.

Example Rees factor

	0	z^2	xz^2	z^3	xz^3
0	0	0	0	0	0
z^2	0	z^2	xz^2	z^3	xz^3
xz^2	0	xz^2	z^2	xz^3	z^3
z^3	0	z^3	xz^3	z^2	xz^2
xz^3	0	xz^3	z^3	xz^2	z^2

Multiplication in the Rees factor S_3/S_4 looks like $\mathbb{Z}_2 \times \mathbb{Z}_2$ with an adjoined zero.

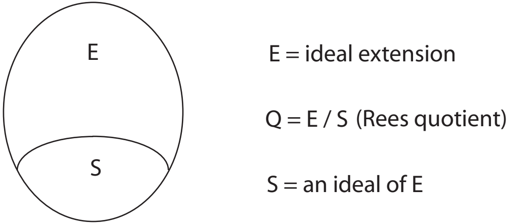
Another Rees factor

	0	z^2	xz^2	z^3	xz^3
0	0	0	0	0	0
z^2	0	0	0	0	0
xz^2	0	0	0	0	0
z^3	0	0	0	0	0
xz^3	0	0	0	0	0

Multiplication in the Rees factor S_4/S_5 is a null semigroup.

Ideal extension

An *ideal extension* of a semigroup S by a semigroup Q with zero is a semigroup E such that S is an ideal of E and Q is the Rees quotient $Q = E/S$.



Semilattices & Archimedean semigroups

- An *semilattice* (short for *lower semilattice*) is a partially ordered set in which any two elements a and b have a greatest lower bound (or *meet*, or *infinum*) $a \wedge b$.
- A commutative semigroup S is *archimedean* when there exist for every $a, b \in S$ some $n \geq 0$ and $t \in S$ such that

$$a^n = tb.$$

- An archimedean semigroup contains at most one idempotent.
- We call an archimedean semigroup is *complete* when it has an idempotent.
- Finite c.s.'s are complete.

Decomposition into three archimedean components

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{e, x, a, xa\}$
null	$\{z, xz, za, xza\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{z^2, xz^2, z^3, xz^3\}$
null	$\{b, xb, zb, xzb\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{b^2, xb^2, zb^2, xzb^2\}$

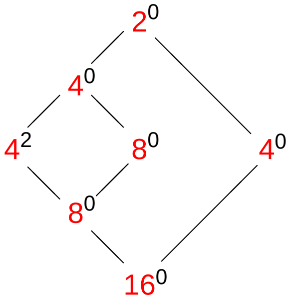
4^0

4^4

4^4

Maximal semilattice homomorphic image

Maximal subgroups in misere Kayles (**0.77**)



Normal quotient order = 16

Misere quotient order = 48

Normal Kayles

	1	2	3	4	5	6	7	8	9	10	11	12
0+	1	2	3	1	4	3	2	1	4	2	6	4
12+	1	2	7	1	4	3	2	1	4	6	7	4
24+	1	2	8	5	4	7	2	1	8	6	7	4
36+	1	2	3	1	4	7	2	1	8	2	7	4
48+	1	2	8	1	4	7	2	1	4	2	7	4
60+	1	2	8	1	4	7	2	1	8	6	7	4
72+	1	2	8	1	4	7	2	1	8	2	7	4
84+	1	2	8	1	4	7	2	1	8	2	7	4
96+	...											

Figure: The nim sequence of normal play Kayles.

Quotient order = 16

Misere Kayles

	1	2	3	4	5	6	7	8	9	10	11	12
0+	x	z	xz	x	w	xz	z	xz^2	v	z	zw	t
12+	xz^2	z	zwx	xz^2	s	xz	z	u	wz^2	zw	zwx	wz^2
24+	f	z	g	xwz^2	wz^2	zwx	z	xz^2	g	zw	zwx	wz^2
36+	xz^2	z	xz^2	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
48+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	wz^2	z	zwx	wz^2
60+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	zw	zwx	wz^2
72+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
84+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
96+	...											

Figure: The pretending function of misere Kayles.

Misere quotient order = 48

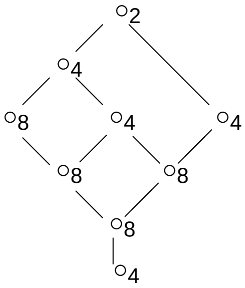
Misere Kayles fortune cookie

Your misere $\mathcal{Q}(\Gamma)$ has order forty-eight.

Your misere and normal $\mathcal{Q}(\Gamma)$'s both have period 12.

Your normal quotient $\mathbb{Z}_2^4$ appears as a misere subgroup.

Dawson's Chess (as **0.07**) at heap 31



$$27P + 101N = 128 \text{ elements} = 50\ (\trianglelefteq \mathbb{Z}_2^3) + 78\ (\text{nilpotent})$$

Normal play period = 34.

Maximum normal play nim heap value $= *7$ (to heap 31)

Quotient order for tame and wild games

Game	Normal	Misere
$\{*0, *1\}$	2	2
$\{*0, *1, *2, *3\}$	4	6
n^{th} tame quotient, $n \geq 2$	2^n	$2^n + 2$
0.123	4	20
0.75	4	8
0.77 (Kayles)	16	48
0.07 (Dawson's Kayles)	16	128+
4.7	4	∞

Computing presentations

Computing presentations: Partial quotients

Calculate, for $n = 1, 2, 3, \ldots$ in turn:

- The partial quotient $\mathcal{Q}_n$
- The partial pretending function $\Phi_n : H_n \to \mathcal{Q}_n$
- The outcome partition $\mathcal{Q}_n = P_n \cup N_n$

where

$$\begin{aligned} H_n &= \{h_1, h_2, \ldots, h_n\} \\ \mathcal{Q}_n &= \mathcal{F}_{H_n}/\rho_n \\ \Phi_n(h_i) &= \text{congruence class of heap of size i} \end{aligned}$$

Compute $(\mathcal{Q}_1, \Phi_1)$, $(\mathcal{Q}_2, \Phi_2)$, $\ldots$ until complete analysis is found.

Partial quotients in **0.123**

$n = 2$, order 2:

$$\langle\, x \mid x^2 = e \,\rangle$$

$n = 7$, order 6:

$$\langle\, x, z \mid x^2 = e,\ z^3 = z \,\rangle$$

$n = 8$, order 12:

$$\langle\, x, z, a, b \mid x^2 = e,\ a^2 = e,\ ab = zb,\ b^2 = xb,\ z^2 = xb \rangle$$

$n > 8$, order 20:

$$\langle x, z, a, b \mid x^2 = a^2 = e,\ z^4 = z^2,\ b^4 = b^2,\ abz = b,\ b^3x = b^2,\ z^3a = z^2 \rangle$$

How to know you're done (for octal games)

The following need to happen:

1. Partial quotients stop getting larger, and
2. "Long enough" observed periodicity of the pretending function.

[This is the misere analogue of Guy & Smith]

Fishing: Widening the net for the next partial quotient

Normal play:

$$\begin{aligned} & (G+G) = 0 \\ \Rightarrow\ & (G+G+G) = G \\ \Rightarrow\ & (G+G+G+G) = (G+G) \\ & \textit{etc} \end{aligned}$$

Misere play:

$(G+G)\ \rho\ 0$ almost never

$(G+G+G)\ \rho\ G$ very frequently (and always if tame)

$(G+G+G+G)\ \rho\ (G+G)$ often

$(G+G+G+G+G)\ \rho\ (G+G+G)$ less often

It's easier to fish in a direct product

Rather than looking for this . . .

$$\langle x, z, a, b \mid x^2 = a^2 = e,\ z^4 = z^2,\ b^4 = b^2,\ abz = b,\ b^3x = b^2,\ z^3a = z^2 \rangle$$

. . . look inside semigroups with generators satisfying $\alpha^{k+2}\ \rho\ \alpha^k$

$$\mathcal{R} = \langle x, z, a, b \mid x^2 = a^2 = e,\ z^4 = z^2,\ b^4 = b^2 \rangle$$

Decide whether $\mathcal{R}$ "supports" the desired quotient $\mathcal{Q}$.

- If not, weaken a relation $\alpha^{k+2}\ \rho\ \alpha^k$, and try again.
- "Support" $\equiv$ has a homomorphic image identical to the quotient.
- May need to add generators, too.

Genus symbols

$$\text{genus}(G) = g^{g_0 g_1 g_2 \cdots}$$

where

$$\begin{aligned} g &= \mathrm{G}^+(G) = \text{ normal play Sprague-Grundy number} \\ g_0 &= \mathrm{G}^-(G) = \text{ misere play Sprague-Grundy number} \\ g_1 &= \mathrm{G}^-(G + *2) \\ g_2 &= \mathrm{G}^-(G + *2 + *2) \\ \ldots &= \ldots \end{aligned}$$

Tame or wild

Tame genus = a genus value that arises in Misere Nim

$$\underbrace{0^{120}, 1^{031}}_{\mathbb{Z}_2}, \underbrace{0^{02}, 1^{13}, 2^{20}, 3^{31}, 4^{46}, 5^{57}, \ldots}_{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots}$$

Wild genus = a genus value that doesn't arise in Misere Nim

$$2^{1420}, 0^{057}, 2^{13131420}, \ldots$$

Say $G = \{A, B, C, D\}$ with known genera for A, B, C, and D, (1^{13}, 0^{120}, 0^{420}, 2^{1420}, resp.)

$$\diamond^{\diamond 05313}$$

$$\diamond^{\diamond 14202}$$

$$\begin{aligned}
\text{genus}(A) &= 1^{131313\cdots} \\
\text{genus}(B) &= 0^{120202\cdots} \\
\text{genus}(C) &= 0^{420202\cdots} \\
\text{genus}(D) &= \underline{2^{142020\ldots}} \\
\text{genus}(G) &= 3^{053131\ldots}
\end{aligned}$$

$$\text{genus}(G) = 3^{0531}$$

Compute single heap genera

Step I: Compute genera

	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
10+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
15+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
20+	0^{02}	...			

Figure: $G*$-values for **0.123**

Find where things become wild

Step II: Stop short at heap k, just before first nontame genus value

	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
10+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
15+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
20+	0^{02}	...			

Figure: $G*$-values for **0.123**

Here, k = heap 7, ie, the partial quotient $\mathcal{Q}_7$ is tame.

Get ready to enter the wild: the last tame quotient

Step III: Initialize $Q_7 = 2^{nd}$ tame quotient as the ideal extension

$$\begin{array}{ll} \mathbb{Z}_2 & = \{0^{120}, 1^{031}\} \\ | & \\ (\mathbb{Z}/2\mathbb{Z})^2 & = \{0^{02}, 1^{13}, 2^{20}, 3^{31}\} \end{array}$$

Order = 6. Presentation:

$$\langle\, x, z \mid x^2 = e,\ z^3 = z \,\rangle$$

Start fighting in the wild

Step IV: Tackle the first wild heap

	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
10+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
15+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
20+	0^{02}	$\dots$			

Figure: $G*$-values for **0.123**

It's telling you that $(*2 + *2 + *2)\ \rho\ (*2)$ now fails.
So weaken it to $(*2 + *2 + *2 + *2)\ \rho\ (*2 + *2)$ instead.

General wild step

Step V: General wild step: heap $n+1$

1. Check if an assignment of the quotient to heap n works at heap $n+1$. If so, done; go on to the next heap size.
2. If not, assign a new generator w for the heap of size $n+1$.
3. Guess period = 2 and a series of trial indices 0,1,2,... for w. Drop all multigenerator relations. You've now got a series of candidate semigroups for the new quotient, each of which is a direct product, and each of which you'll consider in turn, as follows:
4. Find implied N- and P-positions for every element of the direct product.
5. Run verification—is the desired quotient a homomorphic image of this direct product?
6. If not, do some net widening and/or generator introduction. Use structure of normal play and apparent misere mex sets to make guesses. Go back to step 4.
7. If so, find faithful representation by looking at pairwise semigroup elements. Go back to step 1 at next heap size.
8. Continue until (misere) Guy and Smith periodicity condition.

Summary

- There is a misere Sprague-Grundy theory.
- It makes misere play look like normal play.
- Why?

- Problems
  - Dawson's Chess; other games
  - Misere mex mystery
    - Category problem
    - Commutative semigroup literature
    - Complexity gap

References

- More information
  - Taming the Wild in Impartial Combinatorial Games (at arXiv.org)
  - `http://www.plambeck.org/` (click “misere games”)
  - *Software*
    - GAP4
    - Misere Gamesman’s Toolkit (Mathematica)
  - Books
    - Clifford & Preston *Algebraic Theory of Semigroups* (1961)
    - Rosales & Garcia-Sanchez *Finitely generated commutative monoids* (1999)
    - Pierre A. Grillet’s *Commutative Semigroups* (2001)

EXTRA SLIDES FOLLOW

Conjectures

Let $\mathcal{Q}_n$ = a partial quotient of a taking and breaking game.

1. Conjecture 1. $\mathcal{Q}_n$ is finite.
   - True for normal play.
   - $\mathcal{Q}_n$ is *finitely generated*.
   - A theorem of L. Redei implies $\mathcal{Q}_n$ is *finitely presented*.
   - But is it *finite*?
2. Conjecture 2. $\mathcal{Q}_n$ (normal) is a subgroup of $\mathcal{Q}_n$ (misere).
3. Conjecture 3. Every position G in $\mathcal{Q}_n$ satisfies $G^{k+2} \; \rho \; G^k$.
   - False for $\mathcal{Q}$ itself (**4.7**).
4. Conjecture 4. Finite misere quaternary games (octal code digits **0,1,2,3** only) have finite quotients.

Retraction

- When S is a subsemigroup of E, a *retraction* of E onto S is homomorphism of E into S which is the identity on S.
- Every ideal extension E of a monoid S has a retraction

$$a \mapsto ea = ae,$$

where e is the identity element of S.
- An ideal extensions E of S by Q is a *retract ideal extension* when there exists a retraction ψ of E onto S.

Retract ideal lemma

Every ideal extension E of a monoid S by a semigroup Q with zero is a retract ideal extension and is determined by a partial homomorphism ψ of $Q \setminus \{0\}$ into S, namely $\psi : a \mapsto ea$, where e is the identity of S.

Retract ideal extensions

Let S be a semigroup and Q be a semigroup with zero such that $S \cap Q = \emptyset$. If ϕ is a partial homomorphism of $Q \setminus 0$ into S, then the disjoint union $E = S \cup (Q \setminus 0)$, with the multiplication

$$\begin{aligned} a * b &= ab \in Q \quad \text{if } ab \neq 0 \text{ in } Q, \\ a * b &= \phi(a)\phi(b) \quad \text{if } ab = 0 \text{ in } Q, \\ a * y &= \phi(a)y, \\ x * b &= x\phi(b), \\ x * y &= xy \in S \end{aligned}$$

for all $a, b \in Q \setminus 0$ and $x, y \in S$, is a retract ideal extension of S by Q, and every retract ideal extension of S by Q can be constructed in this fashion. Moreover, E is commutative if and only if S and Q are commutative.

Misere quotient example

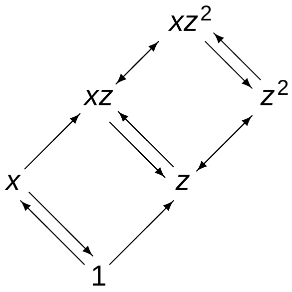


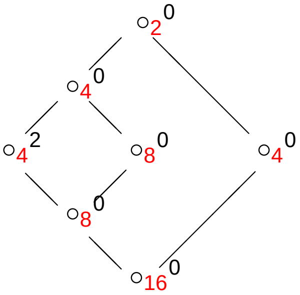
Figure: The action of the generators x (the doubled rungs of the ladder) and z (the southwest-to-northeast-oriented arrows) in $\mathcal{Q}_{\mathbf{0.333}}$.

Questions & Topics

1. How to calculate misere quotient presentations?
2. Do such misere solutions exist in general?
3. How to calculate the single heap equivalents?
4. How to calculate the P- and N- positions?
5. What about that “word reduction” algebra?
    - Knuth-Bendix completion & the confluent term rewriting
6. How to verify that such an analysis solves a game?
    - Verification algorithm to heap size n
    - Guy and Smith periodicity for misere play octals
7. How (and why?) does misere play resemble normal play?
    - Rees factors, idempotents, semilattice decomposition

Idempotents in Misere Kayles (**0.77**)

E = set of all idempotents. Define $g \leq f$ (for $g, f \in E$) to mean $gf = fg = g$. Then $\leq$ is a partial ordering:



The natural partial ordering of the seven idempotents of the misere Kayles (**0.77**) quotient.

Misere loves company

- "This recondite analysis may be recommended to any mathematician [...] It is reasonably clear to me that the general case involves an infinity of expanding moduli. The results cannot be taken very far." TR Dawson, *Caissa's Wild Roses*, 1935, writing about the game **.137**, ie, Dawson's Chess.
- "The necessary tables [for outcome calculations] would be of astronomical size, even for small [positions]..." Grundy and Smith, 1956
- "It would become intolerably tedious to push this sort of analysis much farther, and I think there is no practicable way of finding the outcome [of Grundy's game] for much larger *n*..." Conway, *On Numbers and Games*, 1976
- "You musn't expect any magic formula for dealing with such positions..." Winning Ways, 1st edition, 1983

Too Many Topics, I

- Topic 1: Indistinguishability quotient construction
  - *Definition* of the construction
  - *Existence* of finitely presented misere quotients (to heap n)
  - *Effectiveness* of presentations (Knuth-Bendix & confluence)
  - *Computing* presentations
  - *Proving* quotients correct

Too Many Topics, II

- Topic 2. Relationship to normal play
  - *Recovery* of the Sprague-Grundy theory
  - *Periodicity*—Guy and Smith for misere play octals
  - *Similarity*—of misere and normal periods
  - *The misere mex mystery*—is there a misere mex rule?
- Topic 3. Algebra of misere play
  - *Ideals* and normal play subgroups
  - *Rees congruence* and Rees factors
  - *Idempotents* and their partial ordering
  - *Semilattice* decomposition
  - *Retract ideal extension* and the misere mex mystery
  - *Archimedean components*
  - *Ponizovsky decomposition*

Too Many Topics, III

- Topic 4. Relationship to other misere play techniques
  - *Misere Sprague-Grundy numbers*
  - *Genus computation*
  - *Tame-wild* distinction
  - *Generalized genus sequences* [Allemang 2002]
  - *Canonical forms*
- Topic 5. Particular games
  - *Dawson's* chess
  - *Grundy's* game
  - *Octal* games
  - *Quaternary* games
  - *Non-taking-and-breaking* games
  - *Partisan games*