Misere Play

Following Conway in **ONAG**, we would like to write down some information for each position which will help us to compute outcomes of complicated games. The most obvious bit of information to keep about a game is its own outcome. This would be all the information we would need if we were always given a game as a set of options. However, a game will often appear as a sum of games. Hence, for any game G, we would like to write down the outcomes of $G + x$ for all $x \in X$, X some set of games. What should X be?

We would like X to have the property that if $x \in X$, then $x' \in X$ also. When this holds, we say that X is ***closed under descent.*** If we may assume that we have already computed all outcomes $G' + x \quad (x \in X)$, then this demand guarantees that we will be able to compute easily the outcomes $G + x$ as follows:

In order to compute the outcome of $G + x$ we need to know the outcomes of $G' + x$ and $G + x'$. The former we know by assumption, and, since X is closed, we have already computed the outcome of $G + x'$ (the construction for games allows us to partially order X so that we always compute $G + x'$ before $G + x$).

If we demand that the set X be closed under addition, then whenever we have written down the outcomes of $G + x$ for all $x \in X$, we have also written down, for any $y \in X$, all outcomes $G + y + x, \quad x \in X$.

Theorem I will give us some guidance for choosing what set X to use. Before stating it, I must introduce the set of games we will call ***adders*** (c.f. the game ***adders*** on p. 409 of WW). We will write an adder as $:n$ for $n \geq 0$, and define them by the following formula:

$$:(2n) = \overbrace{*2 + \cdots + *2}^{n \text{ terms}}$$
$$:(2n+1) = \; :(2n) + *1$$

In terms of options, we have the following:

$$:0 = \{\} = *0 = 0$$
$$:1 = \{\, :0\} = *1$$
$$:2 = \{\, :0,\ :1\} = *2$$
$$:3 = \{\, :0,\ :1,\ :2\} = *3$$
$$:4 = \{\, :2,\ :3\}$$
$$:5 = \{\, :2,\ :3,\ :4\}$$
$$\vdots$$
$$:2n+2 = \{\, :2n,\ :2n+1\}$$
$$:2n+3 = \{\, :2n,\ :2n+1,\ :2n+2\}$$

Notice that $:0 = *0 = 0$, and $:1 = *1$.

Theorem I. The three sets $\{:0\}$, $\{:0, :1\}$ and $\{:0, :1, :2, \ldots\}$ (all 'adders') are all closed under both addition and descent. Any other set closed under both of these operations has each of these sets as subsets.

Proof.

The rule for addition tells us that $:0 + :0 = \{\} + \{\} = \{\} = :0$, so $\{:0\}$ is closed under addition. Clearly it is closed under descent.

The rule for addition tells us that

$$:1 + :1 = *1 + *1 = \{*0\} + \{*0\} = \{*1 + *0\} = \{*1\}$$

Now $*1 = \{*0\}$, and $*0$ is an $\mathcal{N}$ position, so $*1$ is a $\mathcal{P}$ position. Hence the move to $*1$ is reversible, whence $\{*1\} = \{\} = 0$, so that $:1 + :1 = :0$, and $\{0, 1\}$ is closed under addition. Clearly it is closed under descent.

For any $n, m \geq 0$,

$$:n + :m = *2 + *2 + \ldots *2 (+ *1) + *2 * 2 + \ldots *2 (+ *1)$$

One summand $+*1$ appears if n is odd, the other appears if m is odd. Since $+$ is associative and commutative, and $*1 + *1 = 0$, at most one summand $+*1$ remains, and the sum is again an adder. This gives us the following rule for summing adders.

For N, M both odd, $:N + :M = :N + M - 2$
For N, M not both odd, $:N + :M = :(N + M)$

I will be making so much use of adders in this work that I will often omit the : when there is no possibility of ambiguity. However, to remind us that the sum for adders is not the same as the sum for the corresponding integers, I will write the sum with a : instead of a $+$, so that $:n + :m$ will be written as $n :m$. Since the adders are closed under addition, and clearly are generated by the adders 1 and 2, we can write the set of adders as $\langle 1, 2 \rangle$.

That the set of adders is closed under descent is clear from the facts that $:2 = \{:0, :1\}$ and $:1 = \{:0\}$.

Let X be a set of games closed under addition and descent. Let s be simplest in $X \setminus \{0\}$. Since X is closed, all options of s are in $\{0\}$, so $s = 0$ or $s = 1$. Let t be simplest in $X \setminus \{0, 1\}$, whence all options of t are in $\{0, 1\}$, so t is one of $\{\} = 0, \{0\} = 1, \{1\} = 0$, $\{0, 1\} = 2$, i.e., $t = 2$. Since X is closed under addition, we have $\langle 1, 2 \rangle \subset X$. ■

The simplicity argument forced our choice of which game to adjoin to the set $\{0\}$ to get $\{0, 1\}$, and again which one to adjoin to get $\{0, 1, 2, 3, \ldots\}$. This fails to determine uniquely which game should be adjoined to the adders to get another set with these properties, since several games (like $\{4, 0\}$, $\{4, 1\}$, $\{4, 2, 1\}$, etc.) can be admitted as simplest members of a larger closed set.

It now seems natural to use the three sets $\{0\}$, $\{0, 1\}$, and $\{0, 1, 2, 3, \ldots\}$ in the role of the set X above. For some game G, if we write down the outcome of $G + x$ for all $x \in \{0\}$, we have only written down the outcome of the game G. If we write the outcome of $G + x$ for all $x \in \{0, 1\}$, we need to write two outcomes. However, since $1 = \{0\}$, we know that G is an option of $G + 1$, so we cannot have that both G and $G + 1$ are $\mathcal{P}$. One of three things is possible, either

i) the outcome of G is $\mathcal{P}$, and $G + :1$ is $\mathcal{N}$