

PRETENDING IN MISÈRE COMBINATORIAL GAMES

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ABSTRACT. We survey results old and new in misère (ie, last-player-losing) impartial combinatorial game theory (CGT). Using *pretending* techniques originally described by Dean Allemang [A1], we obtain complete misère analyses of the octal games **.123**, **.351**, and **.512**. We also solve many nontame (ie, *wild* [WWI], pgs 405–412) subtraction-like *quaternary games*—these are octal games without heap-splitting moves, or (equivalently) the octal games specifiable using the *code digits* [GS] $\{0,1,2,3\}$ only. We correct minor errors in a published solution [A3] for **.53**, but uncover complications with a mistaken solution for **.54** that we haven’t been able to resolve. We close with summaries of current knowledge about **Dawson’s Chess (.137)**, **Guiles (.15)**, and other open problems. In the appendix, we provide alternative, “pretending-centric” proofs of Allemang’s [A3] results on *generalized genus sequences*.

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1. INTRODUCTION

We've spent a lot of time teaching you how to win games by being the last to move. But suppose you are baby-sitting little Jimmy and want, at least occasionally, to make sure you *lose*? This means that instead of playing the normal play rule in which whoever can't move is the *loser*, you've switched to **misère play** rule when he's the *winner*. Will this make much difference? Not *always*...

That's the first paragraph from chapter thirteen ("Survival in the Lost World") of Berlekamp, Conway, and Guy's encyclopedic work on combinatorial game theory (CGT), *Winning Ways for your Mathematical Plays* [WWI]. Because the original Academic Press two-volume work from 1982 is now (early 2003) in the midst of reappearing as a four-volume second edition published by A. K. Peters, we'll have reason to distinguish between [WWI] and [WWII].

And why "not *always*?" The misère analysis of a combinatorial game often proves to be far more difficult than its normal play version. To take a typical example, the normal play of **Dawson's Chess** [D] was solved as early as 1956 by Guy and Smith [GS], but even today, a complete misère analysis hasn't been found. Guy tells the story [Guy91]:

[Dawson's chess] is played on a $3 \times n$ board with white pawns on the first rank and black pawns on the third. It was posed as a *losing* game (last-player-losing, now called **misère**) so that capturing was obligatory. Fortunately, (because we *still* don't know how to play Misère Dawson's Chess) I assumed, as a number of writers of that time and since have done, that the misère analysis required only a trivial adjustment of the normal (last-player-winning) analysis. This arises because Bouton, in his original analysis of Nim [Bou], had observed that only such a trivial adjustment was necessary to cover both normal and misère play...

But even for *impartial* games, in which the same options are available to both players, regardless of whose turn it is to move, Grundy & Smith [GrSm56] showed that the general situation in misère play soon gets very complicated, and Conway [ONAG], (p. 140) confirmed that the situation can only be simplified to the microscopically small extent noticed by Grundy & Smith.

At first sight Dawson's Chess doesn't look like an impartial game, but if you know how pawns move at Chess, it's easy to verify that it's equivalent to the game played with rows of skittles in which, when it's your turn, you knock down any skittle, together with its immediate neighbors, if any.

So misère play can be difficult. But is it a hopeless situation? Returning to chapter 13 in [WWI], one encounters the *genus theory of impartial misère disjunctive sums*, extended significantly from its original presentation in chapter 7 ("How

to Lose When You Must”) of Conway’s *On Numbers and Games* [ONAG]. But excluding the *tame games* that play like Nim in misère play, there’s a remarkable paucity of example games that the genus theory completely resolves. For example, the section “Misère Kayles” ([WWI], pg 411) promises

Although several tame games arise in Kayles (see Chapter 4), wild game’s abounding and we’ll need all our [genus-theoretic] resources to tackle it...

However, it turns out Kayles isn’t “tackled” at all—after an extensive table of genus values to heap size 20, one finds the slightly embarrassing question

Is there a larger single-row P-position?

It was left to the amateur William L. Sibert [SC] to settle misère Kayles using completely different methods. One finds a description of his solution at end of the updated Chapter 13 in [WWII], and also in [SC]; we’ll have even more to say about it here¹. [WWII] summarizes the current situation as follows (pg 451):

Sibert’s remarkable *tour de force* raises once again the question: are misère analyses really so difficult? A referee of a draft of the Sibert-Conway paper wrote “the actual solution will have no bearing on other problems,” while another wrote “the ideas are likely to be applicable to some other games...”

2. BACKGROUND

2.1. Nim: normal vs. misère play. The early history of the game of Nim is shrouded in mystery. Its simple rules invite speculation that it may have been played thousands of years ago. The best hard information seems to come from the first paragraph of Bouton’s 1902 paper “Nim, A Game with a Complete Mathematical Theory,” [Bou], which reads as follows:

The game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory. The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at American colleges, and at some American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

Nim may be played with heaps of beans. Two players move alternately. A move is to choose a heap and remove as many beans from it as you like (perhaps the whole heap, if desired). At least one bean must be taken. Whoever takes the last bean from the final heap wins in normal play; the last player is the *loser* in misère play.

To win at Nim in normal play, make a move so that the nim-sum (\oplus^*) of the numbers of beans in all the heaps becomes zero. To find the nim-sum of numbers,

¹See section 5 (pg 30).

write them in binary and add them without carrying. For example, facing a position with three heaps of size 3, 5, and 1, first calculate the nim-sum

$$3 \overset{*}{+} 5 \overset{*}{+} 1 = 7.$$

A winning move will change this nim-sum to zero. The move $5 \rightarrow 2$ does it:

$$3 \overset{*}{+} 2 \overset{*}{+} 1 = 0.$$

If the nim-sum of a position is already zero, there is no winning move from it. It's then called a Previous player win, or *P-position*. Positions with non-zero nim-sum do always have a winning move for the Next player, and are called *N-positions*. The *P-* and *N-positions* are collectively known as *outcome classes*.

Bouton describes the early history of *misère* Nim as follows:

The [misère] modification of the game [Nim] was described to the writer by Mr. Paul E. More in October 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

Here is the initial part of Bouton's description of *misère* Nim and his proof of its winning strategy:

The game may be modified by agreeing that the player who takes the last counter from the table *loses*. This modification of the three pile game seems to be more widely known than that first described, but its theory is not quite so simple.

A safe combination is defined just as in the first case, *except* that 1, 1, 0 is *not* a safe combination, and 1, 1, 1 and 1, 0, 0 *are* safe combinations. When the first theory indicates that *A* should play 1, 1, 0 he must play either 1, 1, 1 or 1, 0, 0. The earlier part of the proof proceeds as before. In order to complete it, we must show that *B* can never leave 1, 1, 1; that, when 1, 1, 0 is indicated for *A*, he can always play either 1, 0, 0 or 1, 1, 1; and finally that, if the play is carried out this way, *B* must take the last counter...

For example, from the position with heaps of size 6, 1, and 1 in *misère* Nim, the normal play winning move $6 \rightarrow 0$ needs to be reconsidered (it leaves and even number of heaps of size one only). Instead $6 \rightarrow 1$ is the winning move.

That this strategy works for all Nim positions requires proof—the reader is encouraged to supply one! One is tempted to think that all *misère* games might have strategies that involve tracking normal play until the end of the game approaches, and then switching to an end-game strategy similar to the one for *misère* Nim. This is not correct. In fact, the *misère* Nim strategy is a highly fragile object that often doesn't apply to other games. Example settings when it can be generalized to games is the main subject of Ferguson [F2], [F3] and Yamasaki [Y].

2.2. Impartial games: octal games. Two player *impartial games* are characterized by the condition that from every conceivable position, each player would face the same options if he had the move. The impartiality condition rules out games such as Chess, for example, where if Black has the move, he may only move a black piece, and not a white one. The impartial games studied in this paper also involve *complete information* (no details of the position are concealed from either player),

and they will include *no chance moves* (ie, no dice, spinners, etc). In fact, the normal play of an impartial game is always a disguised version of the game of Nim.

What are these other games, and how is it that in normal play they are all disguised versions of Nim? One particularly well studied class of impartial games generalizing Nim are the *octal games* of Guy and Smith [GS]. The class is broad enough to capture games such as Nim and Dawson’s chess, as well as Kayles, “one dimensional tic-tac-toe,” and many others ([WWI], Chapter 4).

Like Nim, octal games can be played with heaps of beans. The rules of an octal game are given by a (possibly infinite) string of base-eight *octal code digits*

$$\mathbf{d_0.d_1d_2d_3d_4\dots}$$

Each code digit d_i governs the circumstances under which it is allowed under the rules of the game to remove i beans from a heap, according to the following table (see [WWI], Chapter 4, or [Guy89], pg 44, for a particularly leisurely presentation):

Value of d_i	Conditions for removal of i beans from a single heap
0	Not permitted
1	If the beans removed are the whole heap.
2	Only if some beans remain as a single heap.
3	Provided the remaining beans, if any, are left in one heap.
4	Only if some beans are left in just two non empty heaps.
5	Provided the remaining beans, if any, are left as two heaps.
6	Only if some beans are left in one or two heaps.
7	Provided the remaining beans are left in at most two heaps.

More generally, (and again quoting from [Guy89], now pg 43):

In general, if in some game we may remove k beans from a heap, provided the rest of the heap is left in exactly a (non-empty) heaps, or b heaps, or c heaps, or \dots (where a, b, c, \dots are distinct), we give that game the code digit $d_k = 2^a + 2^b + 2^c + \dots$ for removal of k beans.

For example, Dawson’s Chess is the game **.137** and Nim is **.333...**

2.3. Grundy numbers.

We review background definitions from [ONAG].

Let $*k$ represent the nim heap of size k . The *Grundy number* of an impartial game position G is the unique number k such that $G + *k$ is a second-player win. Because Grundy numbers may be defined relative to normal or misère play, we distinguish between the *normal play Grundy number* $G^+(G)$ and its counterpart $G^-(G)$, the *misère Grundy number*.

In normal play, Grundy numbers can be calculated using the rules $G^+(0) = 0$, and otherwise, $G^+(G)$ is the least number (from 0,1,2, \dots) that is *not* the Grundy number of an option of G (the so-called *minimal excludant*, or *mex*).

The misère Grundy number is also simple to define (see [ONAG], pg 140, bottom):

$G^-(0) = 1$. Otherwise, $G^-(G)$ is the least number (from 0,1,2, \dots) which is not the G^- -value of any option of G . Notice that this is just like the ordinary “mex” rule for computing G^+ , except that we have $G^-(0) = 1$, and $G^+(0) = 0$.

When normal play is in effect, every game with Grundy number $G^+(G) = k$ can be thought of as the nim heap $*k$. No information about best play of the game is lost by assuming that G is in fact precisely the nim heap of size k . Moreover, in normal play, the Grundy number of a sum is just the nim-sum of the Grundy numbers of the summands. In this sense every normal play impartial game position is simply a disguised version of Nim (see [WWI], Chapter 4, for a full discussion).

2.4. Genera. When misère play is in effect, Grundy numbers can still be defined—as we’ve already said—but many *inequivalent* games are assigned the *same* Grundy number, and the outcome of a sum is *not* determined by Grundy number of the summands. These unfortunate facts lead directly to the apparent great complexity of many misère analyses. Nevertheless progress can be made. We again take the key definition directly from [ONAG], now at the bottom of page 141:

In the analysis of many games, we need even more information than is provided by either of these values [G^+ and G^-], and so we shall define a more complicated symbol that we call the G^* -value, [or *genus*], $G^*(G)$. This is the symbol

$$g^{g_0 g_1 g_2 \dots}$$

where

$$\begin{aligned} g &= G^+(G) \\ g_0 &= G^-(G) \\ g_1 &= G^-(G + *2) \\ g_2 &= G^-(G + *2 + *2) \\ \dots &= \dots \end{aligned}$$

where in general g_n is the G^- -value of the sum of G with n other games all equal to [the nim-heap of size] 2.

At first sight, the genus symbol looks to be an potentially infinitely long symbol in its “exponent.” In practice, it can be shown that the g_i ’s always fall into an eventual period two pattern. By convention, the symbol is written down with a finite exponent with the understanding that its final two values repeat indefinitely.

Evidently the exponent of a genus symbol of a game G is closely related to the outcome of sums of G with all multiples of misère nim heaps of size two. Because such *adder sums* play a key rôle in the sequel, we introduce them next.

2.5. Adders. The *adder* games $:a$ for $a \geq 0$ are defined by setting $:0 = *0$, $:1 = *1$, and $:2 = *2$ (the misère nim heaps of size 0, 1, and 2, respectively). In general

$$:(2a) = \underbrace{:2 + :2 + \dots + :2}_a$$

and

$$:(2a + 1) = :(2a) + :1$$

In terms of options,

$$\begin{aligned}
 : 0 &= \{\} \\
 : 1 &= \{ : 0 \} \\
 : 2 &= \{ : 0, : 1 \} \\
 : 3 &= \{ : 0, : 1, : 2 \} \\
 : 4 &= \{ : 2, : 3 \} \\
 : 5 &= \{ : 2, : 3, : 4 \} \\
 : 6 &= \{ : 4, : 5 \} \\
 : 7 &= \{ : 4, : 5, : 6 \} \\
 : 8 &= \{ : 6, : 7 \} \\
 : 9 &= \{ : 6, : 7, : 8 \} \\
 : 10 &= \{ : 8, : 9 \} \\
 : 11 &= \{ : 8, : 9, : 10 \} \\
 \dots &\quad \dots
 \end{aligned}$$

In terms of addition,

$$\begin{aligned}
 : a + : b &= : (a + b) \text{ if either } a \text{ or } b \text{ (or both) is even} \\
 : a + : b &= : (a + b - 2) \text{ if } a \text{ and } b \text{ are both odd.}
 \end{aligned}$$

In terms of genera:

$$\begin{aligned}
 \text{genus}(: 0) &= 0^{1202\dots} = 0^{120} \\
 \text{genus}(: 1) &= 1^{0313\dots} = 1^{031} \\
 \text{genus}(: 2) &= 2^{2020\dots} = 2^{20} \\
 \text{genus}(: 3) &= 3^{3131\dots} = 3^{31} \\
 \text{genus}(: 4) &= 0^{0202\dots} = 0^{02} \\
 \text{genus}(: 5) &= 1^{1313\dots} = 1^{13} \\
 \text{genus}(: 6) &= 2^{2020\dots} = 2^{20} \\
 \text{genus}(: 7) &= 3^{3131\dots} = 3^{31} \\
 \dots &\quad \dots
 \end{aligned}$$

where the last four entries repeat with period 4. In fact, $:n$ is the exact value of the heap of size n in the game **Adders** (octal code **.73**). See [WWI], pg 409).

2.6. How to lose at 0.123? We'll start at the end of Chapter 13 in [WWI]:

The misère theory of impartial games is the last and most complicated theory in this book. Congratulations if you've followed us so far...

Genus computations, and the nature of the conclusions that can be drawn from them, are what makes Chapter 13 in *Winning Ways* complicated. In this section we illustrate genus computations by using them to initiate the analysis of a particular wild octal game (**.123**).

Since we'll refer to *Winning Ways* frequently, it's useful to have a copy handy in reading the next subsection.

The octal game **.123** can be played with counters arranged in heaps. Two players take turns removing one, two or three counters from a heap, subject to the following additional conditions:

- (1) Three counters may be removed from any heap;
- (2) Two counters may be removed from a heap, but only if it has more than two counters; and
- (3) One counter may be removed only if it is the only counter in that heap.

In *normal play* of **.123**, the last player able to make a legal move is declared the winner. In normal play, each heap size reduces to a nim-heap. The resulting nim sequence² is periodic of length 5, beginning at heap 5.

+	1	2	3	4	5
0+	1	0	2	2	1
5+	0	0	2	1	1
10+	0	0	2	1	1
15+	...				

Table 1

Normal play nim values of **.123**

In *misère play*, the last player to make a legal move is declared to be the *loser* of the game.

Taking our notation again from *Winning Ways* (now Chapter 13, “Survival in the Lost World”), we exhibit the genus sequence of **.123** in Table 2. This sequence is also periodic of length 5:

+	1	2	3	4	5
0+	1	0	2	2	1
5+	0 ⁰²	0	2 ¹⁴²⁰	1 ²⁰	1
10+	0 ⁰²	0	2 ¹⁴²⁰	1 ²⁰	1
15+	...				

Table 2

G*-values of **.123**

In Table 2, a entry that is a simple integer (0, 1, or 2) represents a nim heap of the corresponding size. Nim heaps (and sums of nim heaps) are always *tame* games in misère play. The genus symbols³ of the nim heaps that occur in Table 2 are

$$\begin{aligned}
 0 &= 0^{1202020\dots} = 0^{120} \\
 1 &= 1^{0313131\dots} = 1^{031} \\
 2 &= 2^{2020202\dots} = 2^{20}
 \end{aligned}$$

In misère play of **.123**, the first non-nim-heap occurs at the six-counter heap. It is the game $a = 2_+ = \{2\}$. The eight-counter heap is $b = \{a, 1\}$, and the nine-counter heap is $c = \{a, 0\}$. Although the subsequent games occurring at heap sizes = 1, 3, and 4 (modulo 5) are not identical to a, b, and c, respectively, their respective genera do repeat, as indicated in Table 2.

Here’s what we can (and cannot) do with Table 2:

²See *Winning Ways*, Chapter 4, pg 97, “Other Take-Away Games;” also Table 7(b), pg 104.
³WW pg 402, top.

Single heaps. We can determine the outcome class of *single-heap .123* positions. The first superscript in a heap's genus symbol is 0 if and only if that heap size is a *P*-position. The single heap *P*-positions of **.123** therefore occur at heap sizes

$$1, 5, 6, 10, 11, 15, 16, 20, 21, \dots$$

For example, the heap of size 7 has its first superscript = 1. It is therefore an *N*-position. The winning move is $7 \rightarrow 5$.

Multiple heaps. We cannot immediately determine the outcome class of *multiple-heap .123* positions using Table 2. However, Table 2 does provide a basis for investigating multiheap positions. For example, here is a table that shows the genera of two-heap positions up to heap size nine:

+	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
h_1	0	1	3	3	0	1^{13}	1	3^{0531}	0^{31}
h_2		0	2	2	1	0^{02}	0	2^{1420}	1^{20}
h_3			0^{02}	0^{02}	3	2^{20}	2	0^{420}	3^{02}
h_4				0^{02}	3	2^{20}	2	0^{420}	3^{02}
h_5					0	1^{13}	1	3^{0531}	0^{31}
h_6						0^{02}	0^{02}	2^{20}	1^{13}
h_7							0	2^{1420}	1^{20}
h_8								0^{12}	3^{02}
h_9									0^{02}

Table 3

Some G^* -values of games $h_i + h_j$ in **.123**

Here's how the genus of a particular sum $G = h_8 + h_5$ was computed from earlier values in Table 3. First, we rewrote $\text{genus}(G)$ in terms of its options:

$$\text{genus}(G) = \text{genus}(h_8 + h_5) = \text{genus}(\{h_6 + h_5, h_5 + h_5, h_8 + h_3, h_8 + h_2\})$$

The genus of a non-empty game $G = \{A, B, \dots\}$ can be calculated from the genus of its options A, B, \dots using the *mex-with-($\gamma, \gamma + 1$)-carrying algorithm* (\diamond symbols represent positions with no carry):

$$\begin{aligned} \text{carry}(\gamma) &= \diamond^{05313} \\ \text{carry}(\gamma + 1) &= \diamond^{14202} \\ \text{genus}(h_6 + h_5) &= 1^{131313\dots} \\ \text{genus}(h_5 + h_5) &= 0^{120202\dots} \\ \text{genus}(h_8 + h_3) &= 0^{420202\dots} \\ \text{genus}(h_8 + h_2) &= \underline{2^{142020\dots}} \\ \text{genus}(G) &= 3^{053131\dots} \end{aligned}$$

The result $\text{genus}(G) = 3^{053131\dots} = 3^{0531}$ was computed columnwise, working from left to right. First, the 'base' and 'first superscript' results

$$G^+(G) = \text{mex}(\{1, 0, 0, 2\}) = 3$$

and

$$G^-(G) = \text{mex}(\{1, 1, 4, 1\}) = 0$$

were computed from the corresponding four positions in each option of G , with no carries present. The “carry out” is then $\gamma = 0$. The second superscript result

$$G^-(G + *2) = \text{mex}(\{3, 2, 2, 4, \mathbf{0}, \mathbf{1}\}) = 5$$

involved a similar computation, but with two carry values

$$\{\gamma, \gamma + 1\} = \{0, 1\}$$

thrown into the mex calculation (they’re shown in bold). And so on⁴.

Tame, Restive, Restless or Wild? Another thing we can do with Table 2 is classify the individual heap sizes as *tame*, *restive*, *restless*, or *wild*:

- (1) All the positions equivalent to nim heaps (ie, $h_1 = 1$, $h_2 = 0$, $h_3 = 2$, $h_4 = 2$, $h_{5k} = 1$, and $h_{5k+2} = 0$) are *tame*.
- (2) The game $h_6 = 2_+$ has genus 0^{02} and a single tame option $h_4 = h_3 = 2$. It is therefore⁵ also tame. The positions h_{5k+1} for $k \geq 1$ have genus 0^{02} too, and also happen to be tame, but this is not because they have tame options (they don’t). To show h_{5k+1} is tame, we instead exhibit appropriate *reverting moves*⁶ from h_{5k+1} ’s two possible initial moves. The first of these moves is

$$h_{5k+1} \rightarrow h_{5k+1-2}$$

to a game of genus 1^{20} (with reverting moves to games of genus 0^1 and 0^{02}), and the second is

$$h_{5k+1} \rightarrow h_{5k+1-3}$$

to a game of genus 2^{1420} (with reverting moves to games of genus 0^{02} and 1^0).

- (3) Passing over h_8 for the moment, we turn next to h_9 , of genus 1^{20} . This is a *restive* game⁷. Why? First, it has the correct the genus—for a game G to be restive, its nim values g^γ have to be one of the *restive pairs* in which g is 0 or 1 and γ is 2 or more:

$$0^0, 0^3, 0^4, 0^5, \dots \text{ or } 1^2, 1^3, 1^4, 1^5, \dots$$

There is no further condition if all the options of G are tame⁸. Since these conditions are met by h_{5k+4} for every $k \geq 1$, all these games are restive. What can we conclude from this information about h_{5k+4} ?

- (a) **Sums of a single h_{5k+4} with nim heaps in .123 are tame.** This is an application of the Intermediate Value Theorem⁹.
- (b) **We can easily obtain the genus of such a sum.** If n is a nim heap chosen from $\{0, 1, 2, 3\}$, the genus of $h_{5k+4} + n$ is $(n + 1)^{n+2}$.
- (c) **All even multiples of a heap h_{5k+8} have genus $0^{1202\dots}$.** This is a consequence of the Noah’s Ark Theorem.

⁴See the more complete description of this algorithm in the section titled “*But What if They’re Wild?*” asks the *Bad Child* in [WWI], (page 410). It’s also illustrated on pg 143 in [ONAG].

⁵WW pg 405 top.

⁶WW pg 405 top, but the next paragraph.

⁷WW pg 405, bottom.

⁸WW pg 406.

⁹WW pg 406, bottom.

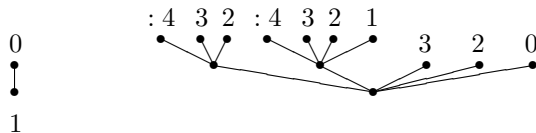


FIGURE 1. Two misère P -positions H for which $H + H$ is an N -position. The game on the left is the nim heap of size 1. It is born on day 1. The game on the right has genus 5^{057} . It is born on day 6, and arises in Kayles (.77) as the sum $1 + 2_2 123$ of two heaps (of sizes 1 and 5, respectively). For both, $\text{genus}(H + H) = 0^{120}$. No games with this property are born on days 2, 3, or 4.

- (4) The genus of h_8 is 2^{1420} , and it is *restless*.¹⁰ In fact, the same is true of all the games h_{5k+8} for $k \leq 1$.

Since h_6 is a tame game of the same genus 0^{02} as the *adder game* $2_2 = :4 = h_3 + h_3$, it's tempting to think that each h_6 can be treated exactly as if it were instead two nim heaps of size two (see section 2.5 for more information on adders).

This would be a mistake, however. The sum $s_1 = h_9 + h_6$ has genus 1^{13} , but $s_2 = h_9 + h_3 + h_3$ has genus 1^{20} . In particular, adding a nim heap of size 1 to s_1 gives a misère P -position, while doing the same to s_2 yields an N -position. This example shows the care that must be taken before drawing any conclusions about sums involving a non-nim-heap tame position such as h_6 and a restive game such as h_9 .

3. PRETENDING

The computations and remarks in the previous section are intended to illustrate the complexities of a misère analysis when the only tools available to be applied are those described in Chapter 13 of *Winning Ways*. Our reader might (rightly) suspect that such computations aren't directly leading to a complete analysis of **0.123**. For this, we jump immediately to the complete solution, which involves *pretending*.

What is pretending? Recall that in the *normal play* of an impartial game, every position H satisfies the identity

$$H + H = 0.$$

Such a normal play relation is in fact an *equation* between game values in the strictest possible sense—it expresses the fact that the sum of two identical games is always a second-player win (P -position) in normal play. But in misère play, there's no guarantee that a sum of identical games will be a second player win—this already fails for the sum $1 + 1$ of two nim heaps of size 1. The same thing can happen for more complicated positions, even when we restrict our attention to games H that are themselves already P -positions (see Figure 1). The outcome of $G + H$ is not a simple function of the outcomes of G and H in misère play (see Figure 2).

¹⁰WW pg 405, bottom.

Form of summands $G + H$	Normal result	Misère result N example	Misère result P example
$N + N$	P iff same Grundy number	$0 + 0 = 0$	$2 + 2 = : 4$
$N + P$	N	$2 + 1 = 3$	$0 + 1 = 1$
$P + P$	P	$1 + 1 = 0$	$: 4 + : 4 = : 8$

FIGURE 2. Misère outcome indeterminacy for sums $G + H$

Nevertheless, we often find that we can usefully *pretend* that weaker identities such as

$$H + H + H = H$$

are valid in misère play. We say “pretend” because for us, such a misère relation will not be interpreted as an *equation* between abstract games values, but rather as a *simplification rule* that happens to *preserve outcomes* in the restricted context of the positions that actually arise as sums within the particular game in question. In certain cases, such *pretending equations* can even lead to a complete analyses of the outcome classes of a wild (ie nontame) game.

We will make the notion of pretending precise in what follows. Readers unsatisfied with our introductory remarks are invited to plunge into the technical details of pretending in section 9.

Before saying more, we consider a specific example.

3.1. Solution to 0.123. To determine **.123** outcome classes, *pretend* that each heap size is either an adder or special game C , A , or B according to Table S1. Its last five values repeat indefinitely. The genera of individual C , A , and B games are given in Table S2.

	1	2	3	4	5
0+	:1	:0	:2	:2	:1
5+	C	:0	A	B	:1
10+	C	:0	A	B	:1
15+	...				

Table S1
Heap pretensions for **.123**

If a position is a sum of at most one C , A , or B and adder terms :a, Tables S1 and S2 can be used to determine whether it is an N- or P-position. For a general **.123** position that contains multiple A’s, B’s, and/or C’s, reduce it to a form where Tables S1 and S2 can be applied using the following additional logic:

C	A	B
0^{02}	2^{1420}	1^{20}

Table S2

Genera for heaps of the form

$$C = 5k + 1, A = 5k + 3, \text{ and } B = 5k + 4, \text{ for } k \geq 1$$

First replace each heap by the equivalent game from Table S1, just as before. Then, depending upon how many B's and C's result, apply exactly one of the two following reduction rules S3 and S4:

- *If there is at least one C or at least two or more B's*, ignore the C terms and reduce the A's and B's to an adder :a according to Table S3:

	even # B's	odd # B's
even # A's	:4	:5
odd # A's	:2	:3

Reduction rule S3

- *Alternatively, if there is no C and at most one B*, reduce the position to at most one A or B and/or adder according Table S4:

	no B's	one B
even # A's	:0	B
odd # A's	A	B + :2

Reduction rule S4

Example 3.1. Suppose a **.123** position G has heaps of size 9, 8, 5, and 3. Is the position a misère P- or N-position, and if the latter, what are the winning move(s)?

Solution: There is no C, one B, and one A present in G . Applying reduction rule S4, we obtain an equivalent

$$B + :2$$

for the heaps 9 and 8 present in G . The remaining heaps of size 5 and 3 in G are equivalent to :1 and :2, respectively. The whole sum is therefore equivalent to

$$(B + :2) + (:1 + :2) = B + :5,$$

whose misère nim value can be read off as 3 from the (infinite) genus symbol $1^{202020\dots}$ for B (the third superscript of this symbol—ie, 2—nim-summed with :1 is 3). Since the misère nim value of G is non-zero, the given position is an N-position. To find a winning move, we must perform similar computations on the eight possible moves from $\{9, 8, 5, 3\}$, looking for moves that result in positions with misère nim value 0.

option	type	example rules applied	reduced form	misère nim value
{7,8,5,3}	{7,A,5,3}	S1/S2	:0 + A + :1 + :2 = A + :3	5
{6,8,5,3}	{C,A,5,3}	S3	:2 + :1 + :2 = :5	1
{9,6,5,3}	{B,C,5,3}	S3	:4 + :1 + :2 = :6	2
{9,5,5,3}	{B,5,5,3}	S1/S2	B + :1 + :1 + :2 = B + :2	0
{9,8,3,3}	{B,A,3,3}	S4	B + :2 + :2 + :2 = B + :6	0
{9,8,2,3}	{B,A,2,3}	S4	B + :2 + :0 + :2 = B + :4	2
{9,8,5,1}	{B,A,5,1}	S4	B + :2 + :1 + :1 = B + :2	0
{9,8,5,0}	{B,A,5,0}	S4	B + :2 + :1 + :0 = B + :3	1

There are three different winning moves from G. They are $8 \rightarrow 5$, $5 \rightarrow 3$, and $3 \rightarrow 1$.

We now describe a simpler and more direct way of performing these calculations. Begin with Table S1, replacing all heaps with the indicated adders or special symbols A , B , or C . Next, apply the *pretending equations*

$$\begin{aligned}
 C + C &= C \\
 A + A + A &= A \\
 B + B + B + B &= B + B
 \end{aligned}$$

to obtain a sum involving at most one C , at most two A 's, and at most three B 's. If at least one C or at least two B 's remain in the sum, then replace all heaps by adders according to the table

	1	2	3	4	5
0+	:1	:0	:2	:2	:1
5+	:4	:0	:2	:5	:1
10+	...				

whose last five values repeat indefinitely to obtain the correct genus. If alternatively the position has no C 's and at most one B , then apply the additional pretension

$$A + A = 0$$

to obtain a position that has either

- (1) Exactly one A , and no B ;
- (2) Exactly one B , and no A ; or finally
- (3) Exactly one B , and exactly one A .

In the first two cases, the genus can be immediately obtained by applying adder shifts to the respective genera of A and B (Table S2). In the final case, one first replaces the sum $A + B$ by $B + :2$, and does the same adder shift computation.

3.2. The general case. Our aim is to describe similarities between various pretending solutions misère octal games. In order to highlight their common features, we'll be stating pretending results for specific games in terms of five key information elements:

- (1) **The Genus Table** presents single-heap genera, arranged in rows of length P , the *global period*. If the analysis is described as *complete* (we also sometimes say the game is *solved*), it can be assumed that the last P values in the genus table repeat themselves indefinitely.

- (2) **The Heap Pretending Table** is similarly arranged in rows with P elements, with the last row understood to repeat indefinitely if the analysis is complete. Each heap in the table is either assigned either adder value $:a$ or a *special game value* of the form $X = X_{p,s}$, reflecting the validity, for the purposes of accurate general computation in this game only, of pretending that in a general sum of heaps, all heaps assigned X satisfy the *pretending equation*

$$\underbrace{X + X + \dots + X}_{p+s \text{ copies}} = \underbrace{X + \dots + X}_s.$$

For example, the symbol $S_{2,0}$ represents the validity of pretending that

$$S + S = 0,$$

i.e., two S heaps can *always* be ignored, no matter what other heaps may occur in the position.

- (3) **The Addition Table** gives the genus of every conceivable *small sum* of special games after they've been fully reduced by pretending. Since each special symbol $X_{p,s}$ that occurs in the heap pretending table gives rise to $p + s$ possible reduced values $0, S, 2 \cdot S, \dots, (p + s - 1) \cdot S$ after pretending is applied to a general position, the adder table is in general of order the product of all these individual $(p+s)$ values.
- (4) **Adder Approximators** collapse much of the information in an addition table. Often the small sums of the special games can be obtained by a further independent assignment of an appropriate *adder approximator value* $:a$ to each special symbol. One obtains the correct genus value for a small sum by computing the genus of the sum of the corresponding adder approximator values, instead.
- (5) **Exceptions** capture information about the particular sums that don't agree with the given adder approximator, if any.

To see the solution of the game **.123** cast into this framework, see section 4.2.

3.3. Local vs global. The correctness of an observed pretending equation ultimately depends on the misère game values that arise “locally” as sums in a particular game. No “global” information about the replaceability of special games in broader contexts can be inferred from a pretending equation. We illustrate this point in this section.

We return to the solution to **.123**. In part, it involves pretending that the heaps h_6 and h_{11} (and in fact h_{5k+1} for all $k \geq 1$) can be treated as if they were a single special game C that behaves as if $C + C = C$. As part of this assertion, we mean that the outcome of $h_6 + T$ is the same as the outcome of $h_{11} + T$ for every game T that *arises as a sum of single-heap positions in .123*.

But what if we expand our attention to include games T that *don't* arise as positions in **.123**? Might not it still be valid to assume that h_6 and h_{11} can be treated as identical games? The answer is no—we can exhibit a game T such that $h_6 + T$ and $h_{11} + T$ have different outcomes.

The reduced game trees for h_6 and h_{11} are shown in Figures 3 and 4. It's already apparent from these diagrams that h_6 and h_{11} are not identical, but exactly what game T distinguishes between them? Such a T won't be a position of **.123**—if it were, our pretending solution would be incorrect.



FIGURE 3. The heap h_6 in **.123** is $2_+ = \{2\}$, a game of genus 0^{02} .

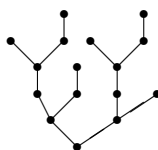


FIGURE 4. The heap h_{11} in **.123**, also of genus 0^{02} .

So what does such a game T look like? One game—obtained via computer calculation—that distinguishes h_6 and h_{11} is

$$T = \{2_+3, 2_+20, 3, 1\},$$

a game of genus 0^{20} . See Figure 5. We have

$$\text{genus}(h_6 + T) = 0^{20}, \text{ while}$$

$$\text{genus}(h_{11} + T) = 0^{0520}.$$

In particular, the sum $h_6 + T$ is a misère N-position, while $h_{11} + T$ is a P-position.

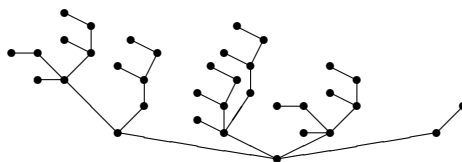


FIGURE 5. The non-**.123** position T distinguishes between h_6 and h_{11} .

4. SOLUTIONS, PART I: PRETENDING

We're ready to state results. What impartial games *have* been solved in misère play? We collect them together here.

4.1. Flanigan's games: .34 and .71. Solutions for the octal games **.34** and **.71** were found by Jim Flanigan. They are described in [WWI] and are reproduced (pg 445, top) in [WWII] also, but a typesetting difficulty there makes the exact period of **.34** difficult to read.

	1	2	3	4	5	6	7	8
0+	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	2 ²⁰	0 ¹²⁰	1 ⁰³¹	0 ¹²⁰	3 ³¹
8+	1 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	2 ²⁰	0 ¹²⁰	3 ⁰⁵³¹	0 ¹²⁰	3 ³¹
16+	1 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	2 ²⁰	0 ¹²⁰	3 ⁰⁵³¹	0 ¹²⁰	3 ³¹
24+	1 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	...				

FIGURE 6. G^* -values for **0.34**

4.1.1. **0.34**. Figure 6 shows the single heap genera for **.34**.

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The game **.34** is particularly unusual amongst the misère octal games. Its fully-reduced single-heap game values are not tame, but are still periodic, of length eight. (The *genera* of such positions often repeat, but almost never their exact *game values*). A far more common situation when nontame positions arise is that fully-reduced game values get increasingly complicated as the heap size gets larger.

Flanigan noticed that the repeating values of **.34** to fall into the period 8 pattern

7	8	9	10	11	12	13	14	...
0	3	1	2 ₂ 1	1	2	0	(2 ₂ 1) + 1	...

after the 6 exceptional initial values 1, 0, 1, 2, 0, 1 [WWI].

Figure 7 expresses the same result using a pretension table. The adder approximator

A	B
: 0	: 1

gives the correct genus for the four sums $i \cdot A + j \cdot B$ ($0 \leq i, j \leq 1$) with the exceptions $\text{genus}(A) = 2^{1420}$ and $\text{genus}(B) = 3^{0531}$.

	1	2	3	4	5	6	7	8
0+	: 1	: 0	: 1	: 2	: 0	: 1	: 0	: 3
8+	: 1	$A_{2,0}$: 1	: 2	: 0	$B_{2,0}$: 0	: 3
16+	...							

FIGURE 7. Heap pretensions for **0.34**

A	B	genus
0	0	0 ¹²⁰
0	1	3 ⁰⁵³¹
1	0	2 ¹⁴²⁰
1	1	1 ⁰³¹

FIGURE 8. Addition Table for **0.34**

4.1.2. **0.71**. Jim Flanigan solved this game (see [WWI], pg 425). Figure 9 shows the single heap genera for **.71**.

	1	2	3	4	5	6
0+	1^{031}	2^{20}	1^{031}	0^{120}	1^{20}	0^{120}
6+	1^{031}	0^{1420}	1^{031}	0^{120}	1^{20}	0^{120}
12+	1^{031}	0^{1420}	1^{031}	0^{120}	1^{20}	0^{120}
18+	1^{031}	0^{1420}	...			

FIGURE 9. G^* -values for **0.71**

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	1	2	3	4	5	6
0+	: 1	: 2	: 1	: 0	$R_{2,3}$: 0
6+	: 1	$S_{2,0}$: 1	: 0	$R_{2,3}$: 0
12+	...					

FIGURE 10. Heap pretensions for **0.71**

R	S	genus
0	0	0^{120}
0	1	0^{1420}
1	0	1^{20}
1	1	1^{02}
2	0	0^{02}
2	1	0^{20}
3	0	1^{13}
3	1	1^{13}
4	0	0^{02}
4	1	0^{02}

FIGURE 11. Addition Table for **0.71**

4.2. **0.123**. Figure 12 shows genera for **0.123**.

	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
10+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
15+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
20+	0^{02}	...			

FIGURE 12. G^* -values for **0.123**

	1	2	3	4	5
0+	:1	:0	:2	:2	:1
5+	$C_{1,1}$:0	$A_{2,1}$	$B_{2,2}$:1
10+	...				

FIGURE 13. Heap pretensions for **0.123**

The game **0.123** is the most interesting three-digit quaternary.

C	A	B	genus
0	0	0	0^{120}
0	0	1	1^{20}
0	0	2	0^{02}
0	0	3	1^{13}
0	1	0	2^{1420}
0	1	1	3^{02}
0	1	2	2^{20}
0	1	3	3^{31}
0	2	0	0^{120}
0	2	1	1^{20}
0	2	2	0^{02}
0	2	3	1^{13}
1	0	0	0^{02}
1	0	1	1^{13}
1	0	2	0^{02}
1	0	3	1^{13}
1	1	0	2^{20}
1	1	1	3^{31}
1	1	2	2^{20}
1	1	3	3^{31}
1	2	0	0^{02}
1	2	1	1^{13}
1	2	2	0^{02}
1	2	3	1^{13}

FIGURE 14. Addition Table for **0.123**

The **0.123** adder approximator

$$\frac{C \quad A \quad B}{: 4 \quad : 2 \quad : 5}$$

gives the correct genus for all twenty four sums of A , B , and C in Figure 14 with the following five exceptions:

$$\begin{aligned} \text{genus}(B) &= 1^{20} \\ \text{genus}(A) &= 2^{1420} \\ \text{genus}(A + B) &= 3^{02} \\ \text{genus}(A + A) &= 0^{120} \\ \text{genus}(A + A + B) &= 1^{20} \end{aligned}$$

4.3. **0.351.** Figure 15 shows genera for **0.351**.

	1	2	3	4	5	6	7	8
0+	1	2	1	2	0 ⁰²	1 ²⁰	0 ⁰²	2 ²⁰
8+	1 ⁰²	2 ²⁰	1 ⁰²	2 ²⁰	0 ⁰²	1 ²⁰	0 ⁰²	2 ²⁰
16+	1 ⁰²	2 ²⁰	1 ⁰²	2 ²⁰	0 ⁰²	1 ²⁰	0 ⁰²	2 ²⁰
24+	1 ⁰²	2 ²⁰	1 ⁰²	2 ²⁰	0 ⁰²	1 ²⁰	0 ⁰²	2 ²⁰
32+	...							

FIGURE 15. G^* -values for **0.351**

	1	2	3	4	5	6	7	8
0+	:1	:2	:1	:2	:4	$A_{2,2}$:4	:2
8+	$B_{2,1}$:2	$B_{2,1}$:2	:4	$A_{2,2}$:4	:2
16+	...							

FIGURE 16. Heap pretending table for **0.351**

Summary

Normal play status: Purely periodic of length 8 [WWI], pg 106.
 Misère play status: Solved 14 March 2003.

Discussion

To determine outcome classes in **0.351**, replace each heap of a general position by adders $:a$ or special games A and B according to the heap pretending table in Figure 16, whose second row repeats itself indefinitely.

Applying the pretending table amounts to reducing four or more occurrences of A 's or 3 or more occurrences of B 's according to

$$A + A + A + A = A + A$$

and

$$B + B + B = B.$$

After the pretending table is applied, the result is a sum of at most three terms—a single adder $:a$, at most 3 A 's, and at most 2 B 's.

Finally, look up the genus of a particular sum of A 's and B 's from this addition table

	$0 \cdot A$	$1 \cdot A$	$2 \cdot A$	$3 \cdot A$
$0 \cdot B$	0 ¹²⁰	1 ²⁰	0 ⁰²	1 ¹³
$1 \cdot B$	1 ⁰²	0 ²⁰	1 ¹³	0 ⁰²
$2 \cdot B$	0 ⁰²	1 ²⁰	0 ⁰²	1 ¹³

FIGURE 17. Genera for small sums of A 's and B 's in **0.351**

and complete the determination of the outcome class by shifting by the adder $:a$.

Another simplification is possible. The entries in table in Figure 17 can be obtained in a simpler fashion by using an *adder approximator* that contemplates that A and B are both $:5$, a game of genus 1^{13} . The genera of arbitrary “approximator sums” of $:5$ are easily calculated using adder arithmetic. When is the approximator not exact? Of the twelve sums in Figure 17 (also reproduced in Figure 19), only four entries shown in Figure 18 break the pattern suggested the approximator:

Position	True Genus	Adder Approximator
A	1^{20}	1^{13}
B	1^{02}	1^{13}
$A + B$	0^{20}	0^{02}
$A + 2B$	1^{20}	1^{13}

FIGURE 18. Exceptions for the **0.351** adder approximator

A	B	genus
0	0	0^{120}
0	1	1^{02}
0	2	0^{02}
1	0	1^{20}
1	1	0^{20}
1	2	1^{20}
2	0	0^{02}
2	1	1^{13}
2	2	0^{02}
3	0	1^{13}
3	1	0^{02}
3	2	1^{13}

FIGURE 19. Addition Table for **0.351**

4.4. **0.512.** Figure 20 shows genera for **0.512**.

	1	2	3	4	5	6
0+	1^{031}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}
6+	1^{031}	0^{02}	1^{431}	2^{20}	2^{20}	2^{20}
12+	1^{13}	0^{02}	1^{431}	2^{20}	2^{20}	2^{20}
18+	1^{13}	0^{02}	1^{13}	2^{20}	2^{20}	2^{20}
24+	1^{13}	0^{02}	1^{13}	2^{20}	2^{20}	2^{20}
30+	...					

FIGURE 20. G^* -values for **0.512**

	1	2	3	4	5	6
0+	: 1	: 1	: 1	: 2	: 2	: 2
6+	: 1	: 4	$A_{2,2}$: 2	: 2	: 2
12+	: 5	: 4	$A_{2,2}$: 2	: 2	: 2
18+	: 5	: 4	: 5	: 2	: 2	: 2
24+	...					

FIGURE 21. Heap pretensions for **0.512**

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Normal Play Status: Period six after two exceptional values: **1,1,1,2,2,2,1,0** (periodic portion bold).

Misère Play Status: Solved 30 April 2003.

To calculate misère outcome classes, it almost always suffices to identify each heap with an adder according to the following table:

	1	2	3	4	5	6
0+	: 1	: 1	: 1	: 2	: 2	: 2
6+	: 1	: 4	<u>: 5</u>	: 2	: 2	: 2
12+	: 5	: 4	<u>: 5</u>	: 2	: 2	: 2
18+	: 5	: 4	: 5	: 2	: 2	: 2
24+	: 5	: 4	: 5	: 2	: 2	: 2
30+	...					

FIGURE 22. Table for computing genera of **0.512** positions

The only exceptions are positions G that are found to have exactly one heap of size 9 or 15 (the underlined values in the second table). If G is such a position, the genus of the 9 or 15 heap should be replaced by the value 1^{431} . All the other heaps in G , if any, can then be treated as adders as indicated in the “adder approximator” table Figure 22.

	1	2	3	4	5	6	7	8	9
0+	1 ⁰³¹	1 ⁰³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	0 ⁰²	2 ²⁰	2 ²⁰	4 ⁴⁶
9+	0 ⁰²	1 ⁶³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	1 ¹³	2 ²⁰	2 ²⁰	4 ⁴⁶
18+	1 ¹³	1 ⁶³¹	2 ²⁰	2 ²⁰	1 ¹³	1 ¹³	2 ²⁰	2 ²⁰	4 ⁴⁶
27+	1 ¹³	1 ¹³	2 ²⁰	2 ²⁰	1 ¹³	1 ¹³	2 ²⁰	2 ²⁰	4 ⁴⁶
36+	1 ¹³	...							

FIGURE 23. G^* -values for **0.53**

	1	2	3	4	5	6	7	8	9
0+	:1	:1	:2	:2	:1	:4	:2	:2	$A_{2,1}$
9+	:4	$B_{2,2}$:2	:2	$C_{2,2}$:5	:2	:2	$A_{2,1}$
18+	:5	$B_{2,2}$:2	:2	:5	:5	:2	:2	$A_{2,1}$
27+	:5	:5	:2	:2	:5	:5	:2	:2	$A_{2,1}$
36+	...								

FIGURE 24. Heap pretending table for **0.53**

4.5. **0.53**. Autogenerated 2003, 4, 8, 9, 36, 14

Summary:

Normal play: The nim sequence is periodic of length 9.

Misère play: Allemang [A1] discovered the solution, but the description of it at the top of pg 551 in [A3] contains three minor errors:

- (1) The text reading “if there are two heaps of 6 items you can pretend...” should read “if there are two heaps of 6 *or more* items you can pretend...”
- (2) The sum 11+14 is a misère N-position, not a P-position. Computing $\text{genus}(11 + 14)$ by looking up values from the first table on pg 551 in [A3], one gets the wrong answer

$$\text{genus}(11 + 14) = \text{genus}(:5 + :5) = \text{genus}(:8) = 0^{02}.$$

The correct genus of 11+14 is 0^{620} , not 0^{02} .

- (3) The behavior of sums involving heaps of size $9k$ —listed as $*4$'s in the table at the top of pg 511—is not fully described. As the pretending table (Figure 24) indicates, these heaps can be collected together and reduced by the pretending equation

$$A + A + A = A.$$

After pretending, either 0, 1, or 2 heaps of (various) sizes $9k$ will remain. If the number is zero, the genus contribution of the original combined A terms is 0^{120} ; if one, 4^{46} , and if two, 0^{02} . One can use the adder approximator

$$\frac{A \quad B \quad C}{*4 \quad :5 \quad :5}$$

to compute the correct genus of an arbitrary sum of A 's, B 's, and C 's, noting the exceptions $\text{genus}(14) = 1^{031}$, $\text{genus}(11) = 1^{631}$, and $\text{genus}(11 + 14) = 0^{620}$. For example, $\text{genus}(9 + 11) = 5^{57}$, which can be calculated by taking the genus of A (which is 4^{46}) and shifting it by $:5$. We belabor this point about the games that “look like” $*4$ in **.53** because it is a source of difficulty in the game **.54**. In that game, heaps that “look like” $*4$ are far from as well-behaved, as we'll see in the section 7.3.

4.6. **0.72**. Dean Allemang’s solution to this game appears in both [A1] and [A3]. As usual, we present the solution as pretending table with adder approximator and a list of exceptions.

	1	2	3	4
0+	1^{031}	0^{120}	2^{20}	3^{31}
4+	1^{031}	0^{120}	2^{20}	3^{31}
8+	1^{431}	0^{02}	2^{20}	3^{31}
12+	1^{431}	0^{02}	2^{1420}	3^{2031}
16+	1^{431}	0^{02}	2^{1420}	3^{2031}
20+	1^{431}	0^{02}	2^{1420}	3^{2031}
24+	1^{431}	0^{02}	2^{1420}	3^{2031}
28+	1^{431}	0^{02}	...	

FIGURE 25. G^* -values for **0.72**

Autogenerated 2003, 4, 10, 13, 35, 42

	1	2	3	4
0+	:1	:0	:2	:3
4+	:1	:0	:2	:3
8+	$A_{2,2}$:4	$D_{2,2}$:7
12+	$A_{2,2}$:4	$E_{2,2}$	$G_{2,2}$
16+	$C_{2,2}$:4	$F_{2,2}$	$G_{2,2}$
20+	...			

FIGURE 26. Heap pretending table for **0.72**

A, C	D, E, F	G, H
:5	:6	:7

FIGURE 27. Adder approximator for **.72**

The adder approximator in Figure 27 has the exceptions listed in Figure 28.

+	A	D, E
16, 20, 24, ...	2^{4620}	1^{631}

FIGURE 28. Exceptions for the **.72** adder approximator

5. SOLUTIONS, PART II: SIBERT-CONWAY DECOMPOSITIONS

The surprising solution to the misère version of Kayles (octal code **.77**) was discovered by the amateur William L. Sibert in 1973, but it was not published until over seventeen years later. In 1989, Sibert wrote a description of his solution in an unpublished 43 page document entitled *The Game of Misere Kayles: The “Safe Number” vs “Unsafe Number” Theory* [Si]. Sibert wrote the following “Preamble” to this document:

Some years ago (about 1961) I was stumped by a problem in an old puzzle book (possibly by Dudeney). It described the plight of a group of tourists in the Alps who were consistently defeated by a young Swiss miss at a pluck-the-petals-from-the-daisy type of game.

Two players took turns plucking petals, and the game required each player to take either one or two petals at a time—with the proviso that if two petals were taken, they had to be physically adjacent.

The winner was the player who took the last petal, and, according to the author, the young lass always won—whether she played first or second.

The “solution” given in the back of the book described her strategy as one of presenting her opponent with a daisy which had been divided into two identical segments. She then simply matched her opponent’s play each time, using the sector opposite the one into which the opponent had just played.

This struck me as an unsatisfactory answer, in that she had to rely on inept play by her opponent in those cases where she played first.

This led me, for some reason, to try and find the winning strategy for a re-defined game in which the wind had randomly blown away a number of petals from a large daisy before the game began.

(It was only much later that I learned that the problem I had set for myself was to find the solution to the well-known [normal play of the] game “Kayles.”)

After many, many, many hours of work, I had the strategy for all possible games in which the largest unbroken string of petals was 168 or less, and was satisfied that this strategy could be applied to any game with a string or strings exceeding 168. (I could have stopped at 166, but did the next two numbers just to round out a final cycle of 12).

The work was done on a commuter train, returning home from work in the evenings, and I used worksheets which bore a crude resemblance to the Grundy scale described in *Winning Ways*, except that my worksheets didn’t “slide.” A copy of one of those worksheets is attached, as Appendix IV.

Having solved the problem, I forgot about it until I happened to see a Martin Gardner column in an issue of *Scientific American* in which he discussed Kayles. The issue came out in 1969 or 1970, and, as I recall, his number values matched mine exactly, except for one number (28?). I rechecked by calculations, and concluded that the variance was almost certainly caused by a “typo” in the article. Many years later, when I acquired a copy of *Winning Ways*, my values were confirmed as correct.

In any case, reading the Gardner article reawakened my interest in Kayles, and I set out to try and solve the problem of the misere version of the game.

In time, I developed the theory of “Safe” and “Unsafe” numbers and by 1973 I had what I believed was a general solution to the game.

Again, I set the matter aside, but for some reason in 1979 I sent an outline of my solution to Mr. Gardner, asking him whether it was correct. He replied that he didn't know, and suggested that I check with Professor Guy.

Once more I let matters slide, but in 1989 I finally sent the "solution" to Professor Guy, and asked for his reaction. The subsequent exchange led me to assemble my work papers into what I trust is a coherent document. What follows is the document which I hope will confirm, under scrutiny, that my theory is correct.

Sibert's manuscript eventually found its way to the desk of John Horton Conway, who found it to be correct, and (with considerable help from Richard Guy!), the paper [SC] appeared.

We'll take up the details of Sibert's solution in section 5.2 (page 34). But first, we'll have an appetizer—the game of *Canapes*, with octal code **.75**.

	1	2
0+	1^{031}	2^{20}
2+	1^{031}	2^{20}
4+	1^{431}	2^{20}
6+	1^{431}	2^{20}
8+	1^{13}	...

FIGURE 29. G^* -values for **0.75**

	1	2
0+	: 1	: 2
2+	: 1	: 2
4+	$A_{2,2}$: 2
6+	$A_{2,2}$: 2
8+	: 5	: 2
10+	...	

FIGURE 30. Heap pretending table for **0.75**

5.1. **Canapes: 0.75.** Autogenerated 2003, 7, 23, 15, 34, 38

Normal play: Nim sequence purely periodic of length two.

Misère play: Solved by Dean Allemang [A1].

When delicious appetizers are arranged in beautiful rows, it's considered impolite to take more than two at a time, or to take exactly two from a row while leaving only one for others. This is the game of *Canapes*, with octal code **.75**.

Canapes is unusual in that its solution can be expressed conveniently using pretending or via a *Sibert-Conway decomposition* [P1].

We'll first give the pretending solution. The heap pretending table is shown in Figure 30. It indicates that the heaps of size 5 and 7 should be treated as identical games to be reduced using the pretending equation

$$A + A + A + A = A + A.$$

After the pretending equation is applied to a general position, at most three A 's remain. Taking the adder approximator $A = :5$ in such a reduced position, we find disagreements between the true genus and the approximator genus *only when exactly one A appears in the position*:

Position	True Genus	Adder Approximator
0	0^{120}	0^{120}
A	1^{431}	1^{13}
$A + A$	0^{02}	0^{02}
$A + A + A$	1^{13}	1^{13}

FIGURE 31. Exceptions for the **0.75** approximator

Now we'd like to give the Sibert-Conway decomposition solution. Suppose for the moment that all the heaps of a Canapes position G are of size 8 or greater. Then the heap pretending table indicates that $\text{genus}(G)$ can be computed by adding up terms each equal $:2$ or $:5$. What possible genus values can be obtained in this way? A glance at section 2.5 reveals that the only possible values for $\text{genus}(G)$ are

$$0^{02}, 1^{13}, 2^{20}, \text{ or } 3^{31}.$$

In particular, we have

$$G^+(G) = G^-(G)$$

for such a G , since the "base" and first superscripts agree in these four possible genus symbols. We're led to the conclusion that when the only heaps that occur in a **.75** position are of size 8 or greater, the *misère outcome is the same as the normal play outcome*. This is a considerable simplification, since the normal play outcome computation is much simpler (all it involves is adding the up *1's and *2's from Figure 29).

What about Canape positions that involve heaps of size smaller than eight tokens? Amongst these, we're concerned only with the heaps of sizes 1, 3, 5 and 7, since the others are $:2$'s in the pretending table, and might as well have been thrown into the argument of the previous paragraph. So which positions involving heaps 1, 3, 5, and/or 7 have outcomes that differ in misère vs normal play? Before writing down the answer, we need some more definitions.

Definition 5.1. Suppose G is an impartial game. We call G an

- **NP Position** if G is an N -position in normal play, but a P -position in misère play;
- **PN Position** if G is an P -position in normal play, but a N -position in misère play;
- **NN Position** if G is an N -position in normal and misère play;
- **PP Position** if G is a P -position in normal and misère play.

A *Sibert-Conway (SC) decomposition* is an explicit classification of each of the positions of a game as either an **NP**-, **PN**-, **NN**-, or **PP**-position.

When writing down particular SC-decompositions, it's often useful to employ subexpressions $E(a, b, \dots)$ or (respectively) $D(a, b, \dots)$ as a shorthand for the set of positions that are made up entirely of an **Even** (respectively, **oDd**) number of

heaps of size a or b or \dots . When single heaps appear with such an expression, we use simple concatenation.

Example 5.2. The NP-positions of Canapes (**.75**) are precisely the positions

$$D(3, 1).$$

The PN-positions are of the form

$$E(3, 1)$$

$$5 D(3, 1), \text{ or}$$

$$7 D(3, 1).$$

For example, $7+3+3+1$ is a PN-position, since it is made up of a single heap of size seven and an odd number of 3's and 1's (when totalled together). Similarly, $3+3+3+3$ is a NP-position, since it has an even number of 1's and 3's. Positions not of the form NP or PN are either PP or NN—for example, $7+7$ is PP , and $10+3$ is NN . We can tell whether they're NN or PP based on their normal play outcome, which is simple to compute using the normal play values inherent in Figure 29.

	1	2	3	4	5	6	7	8	9	10	11	12
0+	1^{031}	2^{20}	3^{31}	1^{031}	4^{146}	3^{31}	2^{20}	1^{13}	4^{046}	2^{20}	6^{46}	4^{046}
12+	1^{13}	2^{20}	7^{57}	1^{13}	4^{64}	3^{31}	2^{20}	1^{031}	4^{64}	6^{46}	7^{57}	4^{64}
24+	1^{731}	2^{20}	$8^{8[10]}$	5^{75}	4^{64}	7^{57}	2^{20}	1^{13}	\dots			

FIGURE 32. G^* -values for **0.77**

5.2. **Kayles: 0.77.** Autogenerated 2003, 6, 7, 9, 3, 19

Normal play: Solved by Guy and Smith [GS]. The nim sequence is periodic of length 12.

Misère play: The solution to misère Kayles was discovered by William Sibert. A writeup appears in the paper by Sibert and Conway [SC], and also in [WWII]. Here is the SC-decomposition:

The PN-positions of Kayles (**.77**) are precisely the positions

$$E(5) E(4, 1),$$

$$E(17, 12, 9) E(20, 4, 1), \text{ or}$$

$$25 E(17, 12, 9) D(20, 4, 1).$$

The NP-positions are of the form

$$D(5) D(4, 1),$$

$$E(5) D(4, 1),$$

$$D(9) E(4, 1),$$

$$12 E(4, 1),$$

$$E(17, 12, 9) D(20, 4, 1), \text{ or}$$

$$25 D(9) D(4, 1).$$

	1	2
0+	1^{031}	2^{20}
2+	1^{031}	2^{20}
4+	1^{431}	2^{20}
6+	1^{431}	2^{1420}
8+	1^{20431}	2^{131420}
10+	$1^{2020431}$	$2^{13131420}$
12+	$1^{202020431}$	$2^{1313131420}$
14+	$1^{20202020431}$...

FIGURE 33. G^* -values for **4.7**

5.3. **Knots (aka Daisies): 4.7.** Autogenerated 2003, 5, 3, 16, 9, 50

Normal play: The nim sequence is purely periodic with period length = 2.

Misère play: Writing about the unusual pattern of values displayed in Figure 33, Berlekamp, Conway and Guy asked:

The first few [genus] values for the three games **.26**, **.57**, and **4.7** are closely related and contain increasing numbers of genus superscripts before settling down. Does this continue?

It does—this game (and its complete analysis via weight functions) is discussed at length in [P1], [A1], and [A3].

To state the solution, we first define a *weight function* on heaps (Figure 34).

n	1	2	3	4	5	6	7	...	m	...
$w(n)$	0	1	0	1	2	3	4	...	$m - 3$...

FIGURE 34. The weight $w(n)$ of a heap of size n in **4.7**

We'll use the notation $[k, n]$ to stand for any position G that has total weight $\leq n$ and normal play Grundy number ("nim heap equivalent") $G^+(G) = k$.

The PN-positions **4.7** are shown in Figure 35, and the NP-positions are shown in Figure 36.

Positions	Nim equivalent	Remarks
$[*0, 0]$	*0	$E(3, 1)$
5 $[*1, 0]$	*0	5 $D(3, 1)$
7 $[*1, 0]$	*0	7 $D(3, 1)$
$(2k + 8) [*2, 2k + 1]$	*3	$k \geq 0$
$(2k + 9) [*1, 2k + 1]$	*3	$k \geq 0$

FIGURE 35. The PN-positions of **4.7**

Positions	Nim equivalent	Remarks
$[*1, 0]$	$*1$	$D(3, 1)$
$(2k + 8) [*1, 2k]$	$*3$	$k \geq 0$
$(2k + 9) [*2, 2k + 1]$	$*3$	$k \geq 0$

FIGURE 36. The NP-positions of **4.7**5.4. Triplicate Knots: **0.157**.

	1	2	3	4	5	6
0+	1^{031}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}
6+	1^{031}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}
12+	1^{431}	1^{431}	1^{431}	2^{20}	2^{20}	2^{20}
18+	1^{431}	1^{431}	1^{431}	2^{1420}	2^{1420}	2^{1420}
24+	1^{20431}	1^{20431}	1^{20431}	2^{131420}	2^{131420}	2^{131420}
30+	$1^{2020431}$	$1^{2020431}$	$1^{2020431}$	$2^{13131420}$	$2^{13131420}$...

FIGURE 37. G^* -values for **0.157**

Autogenerated 2003, 4, 30, 14, 28, 20

Normal play nim sequence purely periodic length = 6.

Misère play: This game is triplicate **4.7** (ie, it is the same as “Knots,” aka “Daisies”, but with each misere position duplicated three times).

6. SOLUTIONS, PART III: QUATERNARY GAMES

A *quaternary game* is a taking-and-breaking game without heap-splitting moves, or, equivalently, an octal game specified using the *code digits* [GS] $\{0, 1, 2, 3\}$ only.

Quaternary games generalize *subtraction games*, which are specified using the octal code digits $\{0, 3\}$ only. It’s known (via Thomas S Ferguson’s “Pairing Property,” see [WWI], page 422) that a heap of size n in *any* misère subtraction game reduces to a nim-heap of size $G^+(n)$, its normal-play nim-value. So every subtraction game is a *tame game* whose misère play is no harder than its normal play.

Many quaternary games have all tame positions also. But others don’t. A quaternary game may contain many wild positions. The game **.123** studied in section 4.2 is a good example.

6.1. **Two digits.** Atop page 424 in [WWI], there’s a table that (in part) classifies the misère play of all the two-digit quaternaries. We duplicate the relevant portion of this table in Figure 38.

	0	1	2	3
.0	.0	.010	.0011	.0110
.1	.10	.110	.i00i	.i100
.2	.0i	.0i	.0i2	.0i2
.3	.i0	M	.i02	.i02

FIGURE 38. The behavior of the two digit quaternaries in misère play.

Fifteen of these table entries show a misère nim sequence and its period (via the superscript dots). The symbol M indicates one game that behaves differently from the others. This special game is called *Stalking*, with quaternary code **.31**. It’s discussed on page 409 in [WWI].

Unlike the other two-digit quaternaries, Stalking does not have all misère nim heaps as values. However, it is still a tame game with eventual period two genus sequence

$$1^{031} 2^{20} 0^{02} 1^{13} 0^{02} 1^{31} 0^{02} 1^{31} \dots$$

6.2. Three digits. Naively, there are $4^3 = 64$ possible three-digit quaternary games **0.d₁d₂d₃** to consider. Of these, sixteen have final code digit zero, and are therefore equivalent to two-digit quaternaries. Of the remaining 48 games, 41 have all misère nim heaps as values, so they reduce to Nim in misère play. The behavior of the remaining seven games is shown in Figure 39. The games **.201** and **.211** happen to *first cousins* of Stalking ([WWI], pgs 101–106).

.121	.123	.131	.201	.211	.312	.331
tame	wild/solved	tame	tame	tame	wild/solved	tame

FIGURE 39. The seven three-digit quaternaries with non-nim heaps in misère play

The games marked “tame” have at least one non-nimheap position, but always tame genera—such games can be played like Nim. Of two marked “wild/solved,” we’ve already studied **.123** in detail in section 4.2. A heap pretending table for **.312** is given in Figure 40.

	1	2
0+	: 1	: 2
2+	: 4	$A_{2,0}$
4+	: 4	: 5
6+	...	

FIGURE 40. Heap pretending for **0.312**

6.3. Four digits. The twenty-one four-digit quaternary games that have at least one wild position are shown in Figure 41. Each entry (**code**, γ , h) shows the first heap size h at which a nontame genus γ occurs.

(.0122, 1²⁰, 12)	(.0123, 1²⁰, 12)	(.1023, 2¹⁴²⁰, 11)	(.1032, 2¹⁴²⁰, 12)
(.1033, 1²⁰, 11)	(.1231, 2¹⁴²⁰, 8)	(.1232, 2¹⁴²⁰, 9)	(.1233, 2¹⁴²⁰, 9)
(.1321, 2¹⁴²⁰, 9)	(.1323, 2¹⁴²⁰, 10)	(.1331, 1²⁰, 8)	(.2012, 1²⁰, 5)
(.2112, 1²⁰, 5)	(.3101, 1²⁰, 4)	(.3102, 0²⁰, 5)	(.3103, 1²⁰, 4)
(.3112, 2¹⁴²⁰, 7)	(.3122, 2¹⁴²⁰, 4)	(.3123, 1³¹, 6)	(.3131, 2¹⁴²⁰, 6)
(.3312, 2¹⁴²⁰, 5)			

FIGURE 41. The twenty-one wild four-digit quaternaries

[To do: incorporate *all* of our results (currently still in *Mathematica*) on 4-digit quaternaries here...]

	1	2	3	4	5	6	7
0+	0^{120}	1^{031}	0^{120}	1^{031}	2^{20}	2^{20}	2^{20}
7+	0^{120}	0^{02}	0^{02}	1^{031}	1^{20}	1^{13}	2^{1420}
14+	0^{120}	0^{02}	0^{02}	1^{031}	1^{20}	1^{13}	2^{1420}
21+	0^{120}	0^{02}	...				

FIGURE 42. G^* -values for **0.0122**

	1	2	3	4	5	6	7
0+	: 0	: 1	: 0	: 1	: 2	: 2	: 2
7+	: 0	$C_{2,1}$	$C_{2,1}$: 1	$A_{2,2}$	$D_{2,1}$	$B_{2,0}$
14+	...						

FIGURE 43. Heap identifications for **0.0122**

6.4. **0.0122.** Autogenerated 2003, 5, 21, 10, 16, 19

Status:

Normal play: Periodic of length 7

Misère Play: Solved 23 May 2003 (the same solution works for **0.0123**):

A heap pretending table for **.0122** is shown in Figure 43. Except for the small special sums

$$\begin{aligned} \text{genus}(A) &= 1^{20} \\ \text{genus}(B) &= 2^{1420} \\ \text{genus}(A+B) &= 3^{02} \end{aligned}$$

the genus of the 54 possible pretending-reduced sums of A, B, C, and D's are given by the adder approximator in Figure 44.

A	B	C	D
: 5	: 2	: 4	: 5

FIGURE 44. Adder approximator for **0.0122**

6.5. **0.1023.** Autogenerated 2003, 5, 25, 9, 43, 3

Normal play: Period 7.

Misère play: Solved 25 May 2003.

This game is closely related to **0.0122**.

Identify heaps with adders :a or special games A, B, C according to Figure 46, whose second row repeats indefinitely:

	1	2	3	4	5	6	7
0+	1 ⁰³¹	0 ¹²⁰	0 ¹²⁰	2 ²⁰	2 ²⁰	1 ⁰³¹	1 ⁰³¹
7+	0 ⁰²	0 ¹²⁰	0 ¹²⁰	2 ¹⁴²⁰	1 ²⁰	1 ⁰³¹	1 ⁰³¹
14+	0 ⁰²	0 ¹²⁰	0 ¹²⁰	2 ¹⁴²⁰	1 ²⁰	1 ⁰³¹	1 ⁰³¹
21+	0 ⁰²	0 ¹²⁰	0 ¹²⁰	2 ¹⁴²⁰	...		

FIGURE 45. G^* -values for **0.1023**

	1	2	3	4	5	6	7
0+	: 1	: 0	: 0	: 2	: 2	: 1	: 1
7+	$C_{2,1}$: 0	: 0	$B_{2,0}$	$A_{2,2}$: 1	: 1
14+	...						

FIGURE 46. Heap pretending for **0.1023**

	1	2	3	4	5	6	7
0+	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	0 ¹²⁰	2 ²⁰	2 ²⁰	2 ²⁰
7+	1 ⁰³¹	0 ⁰²	0 ⁰²	0 ¹²⁰	2 ¹⁴²⁰	1 ¹³	1 ²⁰
14+	1 ⁰³¹	0 ⁰²	0 ⁰²	0 ¹²⁰	2 ¹⁴²⁰	1 ¹³	1 ²⁰
21+	1 ⁰³¹	0 ⁰²	0 ⁰²	0 ¹²⁰	...		

FIGURE 47. G^* -values for **0.1032**

6.6. **0.1032**. Autogenerated 2003, 5, 27, 10, 37, 20

Normal play: Period 7

Misère play: A heap pretending table is shown in Figure 48.

	1	2	3	4	5	6	7
0+	: 1	: 0	: 1	: 0	: 2	: 2	: 2
7+	: 1	$C_{2,1}$	$C_{2,1}$: 0	$B_{2,0}$	$D_{2,1}$	$A_{2,2}$
14+	...						

FIGURE 48. Pretending table for **0.1032**

6.7. **0.1033**. The genus table is shown in Figure 49.

Normal play: Period 7

Misère play: Solved 1 August 2003. The misère game values of **.1033** are a rearrangement of the values for **.1032**—see Figure 50. Swapping the corresponding columns of Figure 48 yields the solution.

6.8. **0.1231**. Solved 1 August 2003. The misère game values of **.1231** are identical to those of **.123**.

6.9. **0.1232**. Autogenerated 2003, 8, 1, 12, 18, 23

Normal play: Period length = 6.

Misère play: Solved 1 August 2003. A pretending table is shown in Figure 52. Figure 53 shows the addition table for wild small sums.

	1	2	3	4	5	6	7
0+	1^{031}	0^{120}	1^{031}	2^{20}	2^{20}	2^{20}	0^{120}
7+	0^{02}	0^{02}	1^{031}	1^{20}	1^{13}	2^{1420}	0^{120}
14+	0^{02}	0^{02}	1^{031}	1^{20}	1^{13}	2^{1420}	0^{120}
21+	...						

FIGURE 49. G^* -values for **0.1033**

The $G_{.1032}$ value at	$7k + 1$	$7k + 2$	$7k + 3$	$7k + 4$	$7k + 5$	$7k + 6$	$7k + 7$
is the $G_{.1033}$ value at	$7k + 3$	$7k + 2$	$7k + 1$	$7k + 7$	$7k + 6$	$7k + 5$	$7k + 4$

FIGURE 50. Translation of **.1032** values into those of **.1033** ($k \geq 0$)

	1	2	3	4	5	6
0+	1^{031}	0^{120}	2^{20}	2^{20}	3^{31}	1^{031}
6+	0^{02}	0^{120}	2^{1420}	2^{20}	1^{2031}	1^{031}
12+	0^{02}	0^{120}	2^{1420}	2^{20}	1^{2031}	1^{031}
18+	...					

FIGURE 51. G^* -values for **0.1232**

	1	2	3	4	5	6
0+	: 1	: 0	: 2	: 2	: 3	: 1
6+	: 4	: 0	$A_{2,3}$: 2	$B_{2,4}$: 1
12+	...					

FIGURE 52. Pretending table for **0.1232**

6.10. **0.1233**. Status: Solved 2 August 2003.

The misère game values of **.1233** are identical to those of **.1232**.

6.11. **0.1321**. Status: Solved 2 August 2003.

For $n \geq 0$, the misère game value of the heap of size $n + 1$ in **.1321** is the same as the value of the heap of size n in **.123**. The heap of size 1 in **.1321** is :1.

6.12. **0.1323**. Status: Solved 2 August 2003.

Normal play: Period 6.

Misère play: The misère game value of a heap of size n in **.1323** is identical to the the value of the heap of size $n + 1$ in **.1323**.

6.13. **0.1331**. Normal play: Period 5

Misère play: Solved 2 August 2003. The misère game values of **.1331** are a rearrangement of the values for **.123**—see Figure 55.

6.14. **0.2012**. Normal play: Period 5

Misère play: Solved 3 August 2003. The misère game values of **.2012** are a rearrangement of the values for **.123**—see Figure 56.

+	$0 \cdot A$	$1 \cdot A$	$2 \cdot A$	$3 \cdot A$	$4 \cdot A$
$0 \cdot B$	0^{120}	2^{1420}	0^{120}	2^{1420}	0^{120}
$1 \cdot B$	1^{2031}	3^{031}	1^{31}	3^{431}	1^{31}
$2 \cdot B$	0^{120}	2^{20}	0^{02}	2^{20}	0^{02}
$3 \cdot B$	1^{31}	3^{13}	1^{31}	3^{13}	1^{31}
$4 \cdot B$	0^{02}	2^{20}	0^{02}	2^{20}	0^{02}
$5 \cdot B$	1^{13}	3^{31}	1^{13}	3^{31}	1^{13}

FIGURE 53. Small sum genera for **.1232**

	1	2	3	4	5
0+	1^{031}	1^{031}	0^{120}	2^{20}	2^{20}
5+	1^{031}	0^{02}	0^{120}	2^{1420}	1^{20}
10+	1^{031}	0^{02}	0^{120}	2^{1420}	1^{20}
15+	...				

FIGURE 54. G^* -values for **0.1321**

The $G_{.123}$ value at	$5k + 1$	$5k + 2$	$5k + 3$	$5k + 4$	$5k + 5$
is the $G_{.1331}$ value at	$5k + 1$	$5k + 5$	$5k + 4$	$5k + 3$	$5k + 2$

FIGURE 55. Translation of **.1331** values into those of **.123** ($k \geq 0$)

The $G_{.2012}$ value at	$5k + 1$	$5k + 2$	$5k + 3$	$5k + 4$	$5k + 5$
is the $G_{.123}$ value at	$5k + 2$	$5k + 5$	$5k + 3$	$5k + 6$	$5k + 9$

FIGURE 56. Translation between **.2012** and **.123** ($k \geq 0$)

6.15. **0.2112**. Normal play: Period 5.

Misère play: Solved 3 August 2003. The misère game values of **.2112** are identical to those of **.2012** (which are in turn a rearrangement of the values for **.123**).

6.16. **0.3101**. Autogenerated 2003, 8, 3, 10, 41, 26

Normal play: Period 2.

Misère play. A heap pretending table is shown in Figure 58.

6.17. **0.3102**. Autogenerated 2003, 8, 3, 11, 1, 52

In the underbrush of the four digit quaternaries lurks a snake. Its name is **.3102**. Unlike the other four-digit quaternaries, **.3102** doesn't seem to want to succumb to a pretending-style analysis. We shall see below that this game does have many interesting and unusual properties. We don't have a complete solution for it, although (with difficulty) we are able to describe the correct strategy for three or fewer heaps of arbitrary size.

Here are the results of three attacks, none wholly successful:

	1	2
0+	1^{031}	2^{20}
2+	0^{02}	1^{20}
4+	0^{02}	1^{13}
6+	0^{02}	1^{13}
8+	...	

FIGURE 57. G^* -values for **0.3101**

	1	2
0+	: 1	: 2
2+	$A_{2,1}$	$B_{2,2}$
4+	$A_{2,1}$	$C_{2,1}$
6+	...	

FIGURE 58. Pretending table for **0.3101**

	1	2	3	4	5
0+	1^{031}	2^{20}	0^{02}	1^{13}	0^{20}
5+	1^{02}	2^{13}	0^{02}	1^{13}	0^{20}
10+	1^{02}	2^{13}	0^{02}	1^{13}	0^{20}
15+	...				

FIGURE 59. G^* -values for **0.3102**

FIRST ATTACK. At first, computational evidence suggested that only 18 possible genera occurred in sums in **.3102**, namely

0^{120} , ie, the genus of 0,

1^{031} , ie, the genus of 1,

and all 16 genera of the form

$$a^{b,b+2}$$

where a, b are chosen from $\{0, 1, 2, 3\}$ and that $+$ sign is nim addition. It is true that these are the only the genera that occur in **.3102** sums, provided one restricts attention to heaps involving at most 21 tokens.

But then, on 23 August 2003, we discovered that the position $P = 9 + 7 + 6$, with 22 tokens, has genus 2^{46} . Whoops.

Figure 60 shows the genera of options of P .

Next, we considered all positions involving at most $X = 32$ tokens, and observed that 28 possible genera occur. They're shown in Figure 61, which, when read left-to-right by rows from the top, gives the "order" in which each new genus first occurs for increasing totals of tokens.

It's likely that there are still other genus values in **.3102** at larger positions.

SECOND ATTACK. Another idea: fix a genus g , one of the 28 shown in Figure 61. For each sum $a_i h_i$ with up to X tokens that has genus g , reduce the a_i modulo 2 to get a vector v of length 32 over Z_2 . Throw all such vectors v into a *parity set* $P(g)$. In other words, $P(g)$ gives us an idea of what the a_i 's of sum that can achieve

Move	to position	has genus
9 → 1	8, 7, 6	3 ²⁰
9 → 5	5, 7, 6	3 ¹³
7 → 6	9, 6, 6	1 ¹³
7 → 3	9, 3, 6	0 ⁰²
6 → 5	9, 7, 5	3 ³¹
6 → 2	9, 7, 2	1 ¹³
	9, 7, 6	2 ⁴⁶

FIGURE 60. The position 9+7+6 has genus 2⁴⁶ in **.3102**

1 ⁰³¹	0 ¹²⁰	2 ²⁰	3 ³¹
0 ⁰²	1 ¹³	0 ²⁰	1 ³¹
1 ⁰²	2 ⁰²	0 ¹³	2 ¹³
3 ¹³	3 ²⁰	3 ⁰²	2 ³¹
0 ³¹	1 ²⁰	2 ⁴⁶	3 ⁵⁷
0 ⁶⁴	1 ⁷⁵	3 ⁴⁶	2 ⁵⁷
1 ⁶⁴	0 ⁷⁵	3 ⁶⁴	2 ⁷⁵

FIGURE 61. The twenty-seven possible genera for a sum in **.3102**, to positions involving at most 32 total tokens.

genus g look like modulo 2. The number of possible v 's is 2^X for each g , although we examined many fewer positions than that—in fact just $P(35)$, the number of partitions of 35, which is 14,883.

It's interesting to construct the 28×28 (symmetric) matrix M with elements i, j (not illustrated here) showing the cardinality of the intersection of $P(g_i)$ with $P(g_j)$. If this matrix were diagonal, it would mean that we could determine the genus of a sum $h_i x_i$ by just looking at the h_i 's modulo 2. (Or maybe not...)

M proves to be a sparse matrix with very few off-diagonal elements. Is this a coincidence, or expected? What is going on here?

THIRD ATTACK. Call a position *frisky* if its misère play outcome differs from its normal play outcome. Then a complete analysis of the frisky positions of **.3102** together with its (easily computed) normal play outcomes would completely determine misère outcomes.

So what positions in **.3102** are frisky?

We can describe the structure of all frisky positions in **.3102** that have at most three heaps, after introducing some more notation. Suppose $\mathbf{p} = \{p_0, p_1, p_2, \dots\}$ is an infinite sequence of multi-heap positions where each p_{i+1} is obtained from p_i by adding one more token to a particular (fixed) heap. For example,

$$\begin{aligned}
 p_0 &= 7 + 23 + \mathbf{23}, \\
 p_1 &= 7 + 23 + \mathbf{24}, \\
 p_2 &= 7 + 23 + \mathbf{25}, \\
 p_3 &= 7 + 23 + \mathbf{26}, \\
 &\dots
 \end{aligned}$$

might be such a sequence. The *frisky indicator* function ι assigns 1 to p_i if it is frisky, and 0 otherwise. Applying ι elementwise to \mathbf{p} we obtain an infinite binary sequence of values

$$\tau = \iota(p_0), \iota(p_1), \iota(p_2), \dots$$

If \mathbf{p} involves at most three heaps in **.3102**, τ can be shown to be eventually periodic of length five. In order to write down such τ values succinctly, we will collect binary values in groups of five and write them as decimal numbers with an bold underline indicating the periodic portion. For example, the sequence

$$\underbrace{0, 0, 0, 0, 1}_1, \underbrace{0, 0, 1, 0, 1}_5, \underbrace{0, 0, 1, 0, 1}_5, \underbrace{0, 0, 1, 0, 1}_5, \dots$$

would be condensed as

$$1 \underline{\mathbf{5}}.$$

We consider subcases according to the number of heaps a position contains. Let h_i represent a heap of size i .

- *Single-heap friskies.* A glance at Figure 59 reveals that the single-heap frisky positions h_t in **.3102** are precisely the heaps of size $t = 5k$ or $t = 5k + 1$.
- *Two-heap friskies.* Suppose h_t is the smaller heap in the two-heap **.3102** position

$$h_t + h_{t+k}$$

for some $k \geq 0$. Then it can be shown that sequence of binary values

$$\tau = \iota(h_t + h_t), \iota(h_t + h_{t+1}), \iota(h_t + h_{t+2}), \dots$$

is eventually periodic length five, for every t . The τ -sequence for each value of t is shown in Figure 62, whose last row repeats indefinitely.

- *Three-heap friskies.* Let

$$h = h_t + h_{t+s} + h_{t+s+k}$$

be a general three-heap **.3102** position specified by its smallest heap size t and arbitrary integers $s, k \geq 0$. We consider subcases according to the value of t modulo 5:

- (1) *Suppose $t = 5m$.* If the smallest heap in a three-heap **.3102** position p is a multiple of five, then p is never frisky.
- (2) *Suppose $t = 5m + 1$.* If $m = 0$, the table shown in Figure 63 applies. Each sequence entry in the table is internally indexed by k . For $m \geq 1$, Figure 64 gives a similar, but different table of τ -values that proves to be *independent* of the chosen value $m \geq 1$.
- (3) *Suppose $t = 5m + 2$.* The values in Figure 65 are periodic of length twenty and are independent of the choice of m .
- (4) *Suppose $t = 5m + 3$.* The τ values are purely periodic of length five. See the table in Figure 66.
- (5) *Suppose $t = 5m + 4$.* The τ values are purely periodic of length five. See the table in Figure 67.

For example, suppose it's desired to find the misère outcome of

$$h = 22 + 50 + 67$$

in **.3102**. Because the smallest heap in h is congruent to 2 modulo 5, Figure 65 applies. We have $s = 28$, so $\tau = \underline{\mathbf{40}}$. The $k = 17$ th value in τ is 0, so $\iota(h) = 0$.

Therefore h is *not* frisky—its misère outcome is the same as its normal play outcome. Consulting Figure 59, the normal play value of h is

$$*2 + *0 + *2 = *0,$$

which is a P -position. So h is also a second-player win in misère play.

t	0	1	2	3	4
0+	16 <u>24</u>	2 <u>18</u>	<u>6</u>	<u>6</u>	<u>0</u>
5+	<u>0</u>	<u>2</u>	<u>6</u>	<u>6</u>	<u>0</u>

FIGURE 62. τ -values for two-heap **.3102** positions. The last (ie second) row repeats indefinitely.

s	0	1	2	3	4
0+	<u>17</u>	<u>0</u>	<u>3</u>	<u>0</u>	<u>12</u>
5+	<u>0</u>	<u>1</u>	<u>3</u>	<u>0</u>	<u>0</u>
10+	<u>12</u>	<u>9</u>	<u>3</u>	<u>0</u>	<u>0</u>

FIGURE 63. τ -values for three-heap **.3102** positions with smallest heap of size 1. The third row repeats indefinitely.

s	0	1	2	3	4
0+	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
5+	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
10+	4 <u>12</u>	<u>8</u>	<u>3</u>	<u>0</u>	<u>0</u>
15+	<u>12</u>	<u>9</u>	<u>3</u>	<u>0</u>	<u>0</u>

FIGURE 64. τ -values for three-heap **.3102** positions with smallest heap value equal to one of 6, 11, 16, ... The fourth row repeats indefinitely.

s	0	1	2	3	4
0+	<u>3</u>	<u>4</u>	<u>4</u>	<u>40</u>	<u>80</u>
5+	<u>3</u>	<u>4</u>	<u>4</u>	<u>40</u>	<u>80</u>
10+	<u>15</u>	<u>5</u>	<u>04</u>	<u>40</u>	<u>4</u>
15+	<u>14</u>	<u>15</u>	<u>04</u>	<u>0</u>	<u>4</u>
20+	10 <u>14</u>	<u>15</u>	<u>4</u>	<u>0</u>	<u>04</u>
25+	10 <u>14</u>	<u>5</u>	<u>4</u>	<u>40</u>	<u>04</u>
30+	<u>14</u>	<u>5</u>	<u>04</u>	<u>40</u>	<u>4</u>

FIGURE 65. τ -values for three-heap **.3102** positions with smallest heap size congruent to 2 modulo 5. The last four rows repeat indefinitely (ie, period twenty).

The behavior of positions involving more than four heaps in **.3102** appears to be even more complicated.

Can these computations be extended somehow to a complete solution to misère **.3102**? This game is still an open problem.

s	0	1	2	3	4
0+	<u>6</u>	<u>6</u>	<u>0</u>	<u>0</u>	<u>2</u>

FIGURE 66. τ -values for three-heap **.3102** positions with smallest heap size equal to one of 3, 8, 13, \dots . The five values repeat indefinitely.

s	0	1	2	3	4
0+	<u>0</u>	<u>0</u>	<u>12</u>	<u>9</u>	<u>3</u>

FIGURE 67. τ -values for three-heap **.3102** positions with smallest heap size equal to one of 4, 9, 14, \dots . The five values repeat indefinitely.

7. UNSOLVED GAMES

	1	2	3	4	5	6	7
0+	0	1	1	2	0	3	1
7+	1	0	3 ¹⁴³¹	3	2 ⁰⁵²⁰	2	4 ¹⁴⁶
14+	0	5 ⁰⁵⁷	2 ⁰⁵²⁰	2	3 ¹⁴³¹	3	0 ⁰²
21+	1 ⁰³¹	1 ¹³	3 ¹⁴³¹	0 ³¹	2 ⁰⁵²⁰	1 ⁴³¹	1 ¹³
28+	0 ¹²⁰	4 ⁰⁵⁶⁴	5 ⁰⁵⁷	2 ²⁰	7 ¹⁴⁸⁷⁵	4 ⁵⁷	0 ⁰²
35+	1 ⁰³¹	1 ¹³	2 ¹⁴²⁰	0 ³¹	3 ⁰⁶³¹	1 ⁴³¹	1 ¹³
42+	0 ¹²⁰	3 ³¹	3 ⁰⁵³¹	2 ²⁰	2 ¹⁷²⁰		
49	...						

FIGURE 68. G^* -values for **0.07**

7.1. Dawson’s Kayles (.07) & Dawson’s Chess (.137). Summary

Normal play status: Periodic of length 34 [GS], see Table 2, below.
 Misère play status: Open. Ferguson [F] gives a complete analysis of positions with no heap larger than size 20. Allemang [A1] goes one heap size farther, to size 21, as we describe below.

Discussion

In an unpublished research note [F], Thomas S Ferguson writes:

Among many imaginative fairy chess problems of T. R. Dawson (1935), Problem 80 asks for the solution of a game that has become to be known as Dawson’s Chess.

Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, n adjacent files, White to play, at losing game, what is the result?

It is understood that a capture must be made when possible.

In terms of removing counters from piles, the rules may be described as follows: (1) a pile consisting of a single counter may be removed, (2) two counters may removed from any pile, and (3) three counters may be removed from any pile and if desired that pile may be split into two parts. This game is a member of the class of octal games of Guy and Smith [GS] — specifically **.137** in their notation. Under the rule that the last player to move wins, Guy and Smith show that the game has a remarkable analysis with a Grundy function [nim sequence] eventually periodic of period 34.

However, under the rule proposed by Dawson that the last player to move loses, the game becomes more difficult to analyze. As a partial analysis, Dawson gave some tentative results.

For small values of n, up to at least 50, first player loses if n equals 1, 2, 6, 7, or 11, modulus 14. In the case of remainder 4 mod 14, the first player wins whatever move he plays first, e.g. for cases 4, 18, and 32 files.

Since a straightforward analysis listing winning listing winning positions becomes exceedingly difficult for values of n beyond 20, one wonders how Dawson, so obviously gifted with combinatorial skills, carried out his analysis to $n = 50$.

In 1974 with the aid of a computer at UCLA, it was discovered that Dawson must have made an error in his analysis, since (1) the first player can win with $n = 43$ by moving the central pawn (eliminating the three central files), and (2) for $n = 32$, the first player can make a losing moving by moving the 5th or 11th pawn from the end.

In this note, we show these two facts without the aid of a computer using the analysis of misère games developed by Conway [ONAG]. We then give a complete analysis of Dawson’s chess when all piles are of size less than 21...

Dawson’s Kayles (.07) and **Dawson’s Chess (.137)** are first cousins—a heap of size n in Dawson’s Kayles behaves the same as the heap of size $n - 1$ in Dawson’s Chess. The game **.4** is a second cousin of Dawson’s Kayles—a heap of size n in **.4** behaves the same as a heap of size $n - 2$ in Dawson’s Kayles. Winning Ways (pg 93) points out that that the nim sequence of **.17** is obtained from those for Dawson’s Kayles by nim-adding 1 when n is odd.

The normal-play nim sequence of Dawson’s Chess, which originally appeared in [GS], also appears on pg 89 in [WWI]:

n	0 1 2 3 4 5	10	20	30
0+	0 1 1 2 0 3 1 1 0 3 3 2 2 4 0 5 2 2 3 3 0 1 1 3 0 2 1 1 0 4 5 2 7 4			
34+	0 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 2 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4			
68+	8 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 9 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4			
102+	8 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 9 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4			
136+	8 1 1 ...			

Table 2: The Remarkable Periodicity of Dawson’s Chess

This sequence is periodic of length 34 after seven exceptional values

$$G(0) = G(34) = G(14) = 0,$$

$$G(16) = G(17) = G(34) = 2,$$

$$G(31) = 2.$$

For misère play, [WWI] (pg 418) tabulates 42 terms of the genus sequence and arranges them in rows of 14 values. We’ve done that in Figure 1, above. Conway [ONAG] gives a table (pg 145) of **.4** misère values to heap size 14. Allemang [A1] gives the “blurry” genus sequence of **.07** (in which all symbols equal to 4 or greater are replaced by a single # symbol) to heap size 48. The genus sequence of **.17** is obtained from by nim adding 1 to every symbol appearing at an odd heap sizes

	1	2	3	4	5	6	7
0+	1	1	0	2	1	3	0
7+	1	1	3^{1431}	2	2^{0520}	3	4^{146}
14+	1	5^{057}	3^{1431}	2	2^{0520}	3	1^{13}
21+	1^{031}	0^{02}	3^{1431}	1^{20}	2^{0520}	0^{520}	1^{13}
28+	...						

FIGURE 69. The G^* -values for **0.17** are closed related to those of **.07**

The misère game values $d[n]$ in Dawson’s chess for $n \leq 21$ display several interesting features.

n	d[n]	Game options in reduced form
1	n[0]	{}
2	n[1]	{n[0]}
3	n[1]	{n[0]}
4	n[2]	{n[0], n[1]}
5	n[0]	{}
6	n[3]	{n[0], n[1], n[2]}
7	n[1]	{n[0]}
8	n[1]	{n[0]}
9	n[0]	{}
10	d[10]	{a[4], n[1], n[2]}
11	n[3]	{n[0], n[1], n[2]}
12	d[12]	{a[5], d[10], n[0]}
13	n[2]	{n[0], n[1]}
14	d[14]	{d[12], d[17], a[4], n[1], n[3]}
15	n[0]	{}
16	d[16]	{d[10]+n[2], d[14], d[19], n[0], n[2]}
17	d[17]	{a[5], d[10], n[0], n[3]} = d[10] + n[1]
18	n[2]	{n[0], n[1]}
19	d[19]	{d[12], d[17], a[4], n[1]} = d[12] + n[1]
20	n[3]	{n[0], n[1], n[2]}
21	d[21]	{d[16]+n[1], d[14], d[19], a[5], d[10], n[2]}

The positions reduce to nim heaps $n[k]$ until heap 10, when the “adder” game $a[4] = 2 + 2 = 2_2$ first appears as an option of $d[10]$. The equations

$$\begin{aligned}
 d[17] &= d[10] + n[1] \\
 d[19] &= d[12] + n[1] \\
 \text{genus}(d[10]) &= \text{genus}(d[19]) \\
 \text{genus}(d[12]) &= \text{genus}(d[17])
 \end{aligned}$$

are striking, as well as the appearance of $d[17]$ and $d[19]$ as fully-reduced *options* of $d[14]$ and $d[16]$, respectively.

Provided the maximum heap size in a Dawson’s Kayles position is less than or equal to 21, Allemang [A1] gives a complete analysis. Replace the non-nim-heap games

with formal symbols

$$a = d[10] = d[19]$$

$$b = d[12] = d[17]$$

$$c = d[14]$$

$$d = d[16]$$

$$e = d[21].$$

Facing a general Dawson's Kayles position with no heap larger than size 21, replace each non-nim-heap using the equations above, then apply the *pretensions*

$$a + a = 0$$

$$b + b = 0$$

$$c + c = 0$$

$$d + d = 0$$

$$e + e = e$$

to reduce the position to a sum involving an adder $a[n]$ and at most one of each symbol a, b, c, d, e . The outcome class (P- or N-position) can then be determined by consulting the *blurry genus table*

e	d	c	b	a	G
0	0	0	0	0	1202
0	0	0	0	1	1#31
0	0	0	1	0	0#20
0	0	0	1	1	0313
0	0	1	0	0	1###
0	0	1	0	1	1###
0	0	1	1	0	0###
0	0	1	1	1	0###
0	1	0	0	0	0###
0	1	0	0	1	0###
0	1	0	1	0	1###
0	1	0	1	1	1###
0	1	1	0	0	0313
0	1	1	0	1	0#20
0	1	1	1	0	1#31
0	1	1	1	1	1202
1	0	0	0	0	0202
1	0	0	0	1	3131
1	0	0	1	0	2020
1	0	0	1	1	1313
1	0	1	0	0	####
1	0	1	0	1	####
1	0	1	1	0	####
1	0	1	1	1	####
1	1	0	0	0	####
1	1	0	0	1	####
1	1	0	1	0	####
1	1	0	1	1	####
1	1	1	0	0	1313
1	1	1	0	1	2020
1	1	1	1	0	3131
1	1	1	1	1	0202

An entry $g_0g_1g_2g_3$ in the G column shows the exponent part of a conventional genus symbol $g^{g_0g_1\dots}$. It's called a *blurry* symbol because all misère nim values greater than or equal to 4 have been replaced by a single symbol #.

Here's how to use the blurry genus table to determine Dawson's Kayles outcome classes provided no heap is larger than size 21, adapted from Allemang's introduction [A1] to his Appendix II:

For an example of how to use this table, suppose that we are playing Dawson's Kayles (octal code .07) and we are presented with the following position:

$$21\ 17\ 12\ 10\ 7\ 4$$

The heaps of sizes 7 and 4 equal nim heaps of sizes 1 and 2 respectively.

Look up 12 and 17. We see that we can pretend that $b + b = 0$, so we can ignore these two heaps.

The heap of size 10 is the game called a , and the heap of size 21 is called e .

We look up the line in the second part of the entry for .07 that has a “1” under the headings a and e , and find that the genus is (3131). Since other two heaps (7 and 4) add up to *3, we find that this sum is a P -position.

At heap 22 and beyond, the positions of Dawson’s game show considerably less simplification. A complete analysis seems to be out of reach using existing techniques.

	1	2	3	4	5	6	7	8	9	10
0+	1^{031}	1^{031}	0^{120}	1^{031}	1^{031}	2^{20}	2^{20}	1^{031}	2^{20}	2^{20}
10+	1^{031}	1^{031}	0^{120}	1^{431}	1^{431}	2^{20}	2^{20}	1^{20}	2^{20}	2^{1420}
20+	1^{431}	1^{431}	0^{120}	1^{431}	1^{431}	2^{20}	2^{1420}	1^{20}	2^{1420}	2^{1420}
30+	1^{13}	1^{631}	0^{120}	1^{431}	1^{631}	...				

FIGURE 70. G^* -values for **0.15**

7.2. Guiles (0.15). Autogenerated 7 March 2003

Summary

Normal play status: Purely periodic of length 10 [GS].

Misère play status: Open. Allemang [A1] analyzes the game completely provided no heap is larger than size 20.

Discussion

Winning Ways [WWI] (pg 93) explains the rules:

The possible moves in the game of Guiles are to remove a heap of 1 or 2 beans completely, or to take two beans from a sufficiently large heap and partition what remains into two smaller heaps. In short, it is the game **.15**.

The normal play nimsequence is purely periodic of length 10, and involves the numbers 0, 1, and 2 only:

1 1 0 1 1 2 2 1 2 2

For misère play, Winning Ways (pg 416) gives the first 32 terms of the genus sequence. Allemang [A1] gives the blurry genus sequence to heap size 64 (in his Appendix I), and also gives tables that allow the the outcome class of general position to be determined provided no heap is larger than 20 (in his Appendix 2).

	1	2	3	4	5	6	7
0+	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	2 ²⁰	2 ²⁰	2 ²⁰	4 ⁴⁶
7+	1 ⁰³¹	1 ¹³	1 ⁶³¹	2 ²⁰	2 ²⁰	2 ²⁰	4 ¹⁵⁷
14+	1 ¹³	1 ¹³	1 ⁴³¹	2 ²⁰	2 ²⁰	2 ²⁰	4 ¹⁵⁷
21+	1 ¹³	1 ¹³	1 ⁴³¹	2 ²⁰	2 ²⁰	2 ²⁰	4 ¹⁵⁷
28+	...						

FIGURE 71. G^* -values for **0.54**

7.3. **0.54.** Autogenerated 2003, 4, 10, 12, 23, 23

Normal play: Periodic, length 7.

Misère play: This game continues to cause headaches. Allemang [A3] contains a solution that isn't completely correct. The game is still open.

For example, in the solution proposed in [A3], heaps of sizes 14 and 21 are treated as identical games both behaving like *4. This is not correct—for example, $\text{genus}(21 + 10) = 5^{046}$, while $\text{genus}(14 + 10) = 5^{64}$. So 21+10 is a misère P-position, while 14+10 is an N-position. It's not even true that $\text{genus}(21 + 14) = 0^{02}$, as one might expect from the proposed solution. In fact, $\text{genus}(21 + 14) = 0^{31}$. It is true that $\text{genus}(14 + 14) = \text{genus}(21 + 21) = 0^{02}$.

More positions of interest are listed in Figures 72, 73, and 74.

k	$\text{genus}(10+7k)$
0	1 ⁶³¹
1, 3, 5, ...	5 ⁵⁷
2, 4, 6, ...	1 ¹³

FIGURE 72. Genera for a single heap of size 10 with piles of size 7 in **.54**

k	$\text{genus}(h+7k)$
0	4 ¹⁵⁷
1	0 ³¹
2, 4, 6, ...	4 ⁴⁶
3, 5, 7, ...	0 ⁰²

FIGURE 73. Genera for a single heap h chosen from $\{14, 21, 28, \dots\}$ with piles of size 7 in **.54**

k	genus($h+7k$)
0	1^{431}
1	5^{075}
2	1^{831}
3, 5, 7, ...	5^{57}
4, 6, 8, ...	1^{13}

FIGURE 74. Genera for a single heap h chosen from $\{17, 24, 31, \dots\}$ with piles of size 7 in **.54**

	1	2	3	4	5	6
0+	1^{031}	2^{20}	0^{02}	1^{20}	0^{02}	2^{20}
6+	1^{02}	2^{20}	0^{02}	1^{20}	0^{02}	2^{20}
12+	1^{02}	2^{20}	0^{02}	1^{20}	0^{02}	2^{20}
18+	1^{13}	2^{20}	0^{02}	1^{13}	0^{02}	2^{20}
24+	1^{13}	2^{20}	0^{02}	1^{13}	0^{02}	2^{20}
30+	1^{13}	2^{20}	0^{02}	1^{13}	0^{02}	2^{20}
36+	1^{13}	2^{20}	0^{02}	1^{13}	...	

FIGURE 75. G^* -values for **0.35**

7.4. **0.35.** Autogenerated 2003, 6, 7, 23, 14, 25

The normal play nim sequence is purely periodic of length 6.

In misère play of **.35**, the first non-nim-heap occurs at the three-counter heap. It is the game $a = 2_+ = \{2\}$. The four-counter heap is $b = \{a, 0\}$, and the five-counter heap is $c = \{b, 3\}$.

Winning Ways contains a tabulation of the single heap genera of **.35** to heap size 18. In Figure 75 these calculations are extended to heap size 40. There seems a period 6 pattern in the genera (starting at heap size 17). Does it continue?

Allemand gives a complete analysis of **.35** positions to heap size 7.

The genus of $h_3 + h_4 + h_6$ is 3^{46} .

Positions up to heap size 3 The correct outcome class (i.e. P- or N-position) for **.35** positions that have no more than three counters per heap can be determined by pretending that the heaps are equivalent to adders according to the following table:

heap	h_1	h_2	h_3
value	:1	:2	:4

Computing **.35** outcome classes to heap size 3

Positions up to heap size 4

At heap size 4, we can pretend that $h_4 + h_4 = 4$.

Positions up to heap size 7

(See Allemand ([A1], pg 48) for the results of computations similar to those described in this section). On 8 June 2003, we found that the following pretending table works for each of the 51653 positions involving at most 40 total tokens and maximum heap size at most seven:

	1	2	3	4	5	6
0+	:1	:2	$C_{1,2}$	$A_{2,2}$:4	$E_{2,1}$
6+	$B_{2,2}$					

Pretending table for **.35** valid to heap size 7

The heaps of size 1 and 2 are equal to :1 and :2 as games, but equating heap 5 with :4 = :2 + :2 is a pretension.

This corresponds to reducing heaps 3, 4, 6 and 7 by the rules:

$$\begin{aligned}
 C + C + C &= C + C \\
 A + A + A + A &= A + A \\
 E + E + E &= E \\
 B + B + B + B &= B + B
 \end{aligned}$$

The corresponding addition table would have $3 \times 4 \times 3 \times 4 = 144$ entries. Can we get by with simpler pretensions? It doesn't seem so—one thing that *won't* work, for example, is to replace the heap 7 pretending equations by the simpler equation

$$B + B + B = B,$$

since $genus(7 + 7 + 7) = 1^{13} \neq 1^{02} = genus(7)$.

Here's what we can say: the adder approximator

symbol	C	A	E	B
heap	3	4	6	7
approximator	:4	:5	:2	:5

Adder Approximator for **.35** to heap 7

gives the correct genus for 127 entries in the addition table, with the following $144 - 127 = 17$ exceptions:

exceptional genus	positions
0^{20}	0101, 0121, 1101, 1121, 2101, 2121
1^{02}	0001, 0021
1^{20}	0100, 0120
2^{02}	0111, 1111, 2111
3^{02}	1011
3^{13}	0011, 0110
3^{46}	1110

Exceptions for the **.35** adder approximator to heap 7. Coefficients on C, A, E and B are shown, so that for example 0121 represents the position A+E+E+B.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0+	1^{031}	1^{031}	1^{031}	0^{120}	1^{031}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}	1^{031}	2^{20}	2^{20}	2^{20}
14+	1^{031}	1^{031}	1^{031}	0^{120}	1^{431}	1^{431}	1^{431}	2^{20}	2^{20}	2^{20}	1^{20}	2^{20}	2^{20}	2^{1420}
28+	1^{431}	1^{431}	1^{431}	0^{120}	1^{431}	1^{431}	1^{431}	2^{20}	2^{20}	2^{1420}	1^{20}	2^{1420}	2^{1420}	2^{1420}
42+	1^{13}	1^{13}	1^{631}	0^{120}	1^{431}	1^{431}	1^{631}	2^{20}	...					

FIGURE 76. G^* -values for **0.115**

7.5. **0.115.** Autogenerated 2003, 4, 30, 9, 50, 19

Normal play nim sequence periodic length = 14

Misère play: Open. In particular, we haven't computed far enough to find the ultimate single-heap genera periodicity, if any.

	1	2	3	4	5	6	7	8	9	10	11
0+	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	2 ²⁰	4 ⁴⁶
11+	1 ⁰³¹	0 ¹²⁰	1 ⁰³¹	1 ⁰³¹	2 ¹⁴²⁰	1 ³¹	2 ²⁰	2 ²⁰	1 ³¹	2 ²⁰	4 ⁴⁶
22+	1 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	1 ⁰⁴⁶	2 ²⁰	1 ³¹	2 ⁵⁷²⁰	2 ²⁰	1 ³¹	2 ²⁰	4 ⁶³¹
33+	1 ⁰³¹	2 ¹⁴²⁰	1 ⁶³¹	1 ⁵⁷	2 ²⁰	1 ³¹	2 ²⁰	...			

FIGURE 77. G^* -values for **0.141**

7.6. **0.141.** Autogenerated 2003, 4, 30, 12, 28, 13

Normal play nim sequence period length = 11, after 13 exceptional values.

Misère play: Open

7.7. **0.144.**

	1	2	3	4	5	6	7	8	9	10
0+	1 ⁰³¹	0 ¹²⁰	0 ¹²⁰	1 ⁰³¹	2 ²⁰	2 ²⁰	2 ²⁰	2 ²⁰	4 ⁴⁶	4 ⁴⁶
10+	1 ⁰³¹	1 ¹³	1 ¹³	1 ⁶³¹	2 ²⁰	2 ²⁰	2 ²⁰	2 ²⁰	4 ⁴⁶	4 ¹⁵⁷
20+	1 ¹³	1 ¹³	1 ¹³	1 ⁶³¹	2 ²⁰	2 ²⁰	2 ²⁰	2 ²⁰	4 ⁴⁶	4 ¹⁵⁷
30+	1 ¹³	1 ¹³	1 ¹³	1 ⁷³¹	2 ²⁰	...				

FIGURE 78. G^* -values for **0.144**

Autogenerated 2003, 4, 30, 10, 55, 52

Normal play nim sequence period length = 10, after 3 exceptional values.

Misère play: Open

7.8. **0.145.**

	1	2	3	4	5	6	7	8	9
0+	1^{031}	0^{120}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}	2^{20}	4^{46}
9+	1^{031}	1^{031}	1^{13}	1^{631}	2^{20}	2^{20}	2^{20}	2^{20}	4^{46}
18+	1^{13}	1^{13}	1^{13}	1^{631}	2^{20}	2^{20}	2^{20}	2^{20}	4^{157}
27+	1^{13}	1^{13}	1^{13}	1^{431}	2^{20}	2^{20}	2^{20}	2^{20}	4^{157}
36+	1^{13}	...							

FIGURE 79. G^* -values for **0.145**

Autogenerated 2003, 4, 30, 12, 19, 37

Normal play nim sequence period length = 9, after two exceptional values.

Misère play: Open

	1	2	3	4	5	6	7	8	9	10	11
0+	1^{031}	1^{031}	0^{120}	1^{031}	1^{031}	2^{20}	2^{20}	2^{20}	2^{20}	2^{20}	4^{46}
11+	1^{031}	1^{031}	1^{13}	1^{431}	1^{631}	2^{20}	2^{20}	2^{20}	2^{20}	2^{20}	4^{157}
22+	1^{13}	1^{13}	1^{13}	1^{431}	1^{431}	2^{20}	2^{20}	2^{20}	2^{20}	2^{1520}	4^{157}
33+	1^{13}	1^{13}	1^{13}	1^{431}	1^{20431}	2^{20}	2^{20}	...			

FIGURE 80. G^* -values for **0.154**

7.9. **0.154.** Autogenerated 2003, 5, 4, 22, 0, 13

	1	2	3	4	5	6	7	8	9	10	11
0+	1 ⁰³¹	1 ⁰³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	1 ⁰³¹	0 ¹²⁰	2 ²⁰	1 ²⁰	4 ⁴⁶	0 ¹²⁰
11+	1 ⁰³¹	1 ⁴³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	1 ⁰³¹	2 ¹⁴⁶	2 ²⁰	1 ²⁰	4 ⁴⁶	2 ¹⁴⁶
22+	1 ⁰³¹	1 ⁶³¹	2 ²⁰	2 ²⁰	1 ⁰³¹	1 ¹³	2 ¹⁴⁶	2 ⁵⁷	1 ²⁰	4 ¹⁴⁶	2 ¹⁴⁶
33+	1 ⁰³¹	1 ⁶³¹	2 ¹⁴²⁰	2 ²⁰	1 ⁰³¹	1 ¹³	2 ¹⁴⁶	...			

FIGURE 81. G^* -values for **0.171**

7.10. **0.171.** Autogenerated 2003, 5, 3, 16, 1, 41

7.11. **0.176.**

	1	2	3	4	5	6	7	8
0+	1^{031}	1^{031}	0^{120}	2^{20}	2^{20}	3^{31}	4^{46}	4^{46}
8+	1^{031}	1^{13}	6^{146}	2^{20}	2^{20}	3^{31}	4^{46}	4^{46}
16+	1^{531}	1^{13}	6^{146}	6^{64}	3^{31}	3^{31}	2^{20}	4^{46}
24+	1^{531}	1^{731}	6^{64}	6^{64}	3^{31}	3^{31}	4^{46}	4^{46}
32+	1^{531}	1^{731}	6^{64}	6^{64}	3^{31}	3^{31}	4^{46}	4^{46}
40+	...							

FIGURE 82. G^* -values for **0.176**

Autogenerated 2003, 5, 3, 16, 29, 31

	1	2	3	4	5	6	7	8	9	10	11	12
0+	1 ⁰³¹	2 ²⁰	0 ⁰²	2 ¹⁴²⁰	1 ⁰³¹	2 ¹⁴²⁰	3 ²⁰³¹	0 ¹²⁰	1 ²⁰³¹	0 ¹²⁰	3 ⁰⁴³¹	0 ¹²⁰
12+	1 ⁰³¹	2 ¹⁶²⁰	3 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	2 ¹⁴²⁰	3 ⁰⁶³¹	0 ¹²⁰	1 ⁰⁵³¹	0 ¹²⁰	3 ⁰⁴³¹	0 ¹²⁰
24+	1 ⁰³¹	2 ¹⁶²⁰	3 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	2 ¹⁴²⁰	3 ⁰⁶³¹	0 ¹²⁰	1 ⁰⁵³¹	0 ¹²⁰	3 ⁰⁴³¹	0 ¹²⁰
36+	1 ⁰³¹	2 ¹⁶²⁰	3 ⁰³¹	2 ¹⁴²⁰	1 ⁰³¹	2 ¹⁴²⁰	3 ⁰⁶³¹	0 ¹²⁰	1 ⁰⁵³¹	0 ¹²⁰	3 ⁰⁴³¹	0 ¹²⁰
48+	1 ⁰³¹	2 ¹⁶²⁰	...									

FIGURE 83. G^* -values for **0.316**

7.12. **0.316.** Autogenerated 2003, 5, 4, 22, 24, 18

Notes added 4 May 2003. Not a bad candidate for further investigation. Its single heap genera seem to settle down quickly.

7.13. **0.375.**

	1	2	3	4	5	6	7	8	9
0+	1 ⁰³¹	2 ²⁰	3 ³¹	1 ⁰³¹	2 ²⁰	4 ¹⁴⁶	3 ³¹	2 ²⁰	1 ¹³
9+	3 ³¹	4 ⁰⁴⁶	2 ²⁰	7 ⁵⁷	4 ⁰⁴⁶	8 ^{68[10]}	1 ¹³	4 ⁸⁴⁶	8 ^{9[10]8}
18+	1 ¹³	2 ²⁰	4 ⁶⁴	8 ^{31[12][14]}	1 ¹³	4 ⁶⁴	3 ³¹	8 ^{9[11]}	1 ²⁰
27+	4 ⁵³¹	8 ⁶⁴	2 ⁴⁶²⁰	1 ^{[12]31}	4 ⁵⁷	8 ⁶⁴	1 ⁷³¹	4 ^{9[11]}	8 ⁶⁴
36+	1 ^{[10][13][10]8}	2 ^{[12]31}	4 ⁶⁴	8 ⁵⁷	...				

FIGURE 84. G^* -values for **0.375**

Autogenerated 2003, 6, 12, 20, 12, 29

We looked at this game from the point of view of Sibert-Conway decompositions in late 2002, and constructed a decomposition to heap size 12 or so.

The nim sequence is periodic of length 18 starting at position 37.

It might be worth looking at the genera above in comparison to Kayles (**.77**).

8. APPENDIX I: A ZOO OF TAME GAMES

8.1. **0.147.**

	1	2	3	4	5	6	7	8
0+	1^{031}	0^{120}	1^{031}	2^{20}	2^{20}	2^{20}	4^{46}	4^{46}
8+	1^{031}	1^{13}	1^{13}	2^{20}	2^{20}	2^{20}	4^{46}	4^{46}
16+	1^{13}	1^{13}	1^{13}	2^{20}	2^{20}	2^{20}	4^{46}	4^{46}
24+	1^{13}	1^{13}	1^{13}	2^{20}	2^{20}	2^{20}	...	

FIGURE 85. G^* -values for **0.147**

Autogenerated 2003, 4, 30, 13, 8, 26

Normal play nim sequence period length = 8, after 2 exceptional values.

Misère play: Tame. But compare **.147** to **.144** and **.145**, which have similar nim sequences but are much more complicated in misère play.

	1	2
0+	1^{031}	2^{20}
2+	1^{031}	2^{20}
4+	0^{02}	2^{20}
6+	0^{02}	2^{20}
8+	0^{02}	2^{20}
10+	0^{02}	2^{20}
12+	...	

FIGURE 86. G^* -values for **0.353**

8.2. **0.353.** Autogenerated 2003, 4, 29, 12, 8, 48

This is a tame game whose single-heap positions can be pretended to be

: 1, : 2, : 1, : 2, : 4, : 2, : 4, ...

in misère play.

8.3. **0.373.**

	1	2	3	4	5	6	7
0+	1^{031}	2^{20}	3^{31}	4^{46}	0^{02}	1^{13}	2^{20}
7+	3^{31}	4^{46}	1^{13}	5^{57}	2^{20}	3^{31}	1^{13}
14+	4^{46}	7^{75}	2^{20}	1^{13}	0^{02}	4^{46}	3^{31}
21+	2^{20}	1^{13}	4^{46}	0^{02}	2^{20}	6^{64}	4^{46}
28+	0^{02}	2^{20}	...				

FIGURE 87. G^* -values for **0.373**

Autogenerated 2003, 4, 29, 13, 0, 55

A tame game. The normal play (and misère) nim sequence is periodic of length 28 after one exceptional value.

8.4. **4.12.**

	1	2	3	4	5	6	7
0+	1^{031}	1^{031}	2^{20}	2^{20}	0^{02}	4^{46}	2^{20}
7+	1^{13}	1^{13}	2^{20}	2^{20}	1^{13}	4^{46}	2^{20}
14+	1^{13}	1^{13}	2^{20}	2^{20}	1^{13}	4^{46}	2^{20}
21+	1^{13}	1^{13}	2^{20}	\dots			

FIGURE 88. G^* -values for **4.12**

Autogenerated 2003, 4, 29, 14, 4, 9

Tame game, period 7.

	1	2	3
0+	1^{031}	2^{20}	4^{46}
3+	1^{13}	2^{20}	4^{46}
6+	1^{13}	2^{20}	4^{46}
9+	1^{13}	2^{20}	4^{46}
12+	1^{13}	2^{20}	...

FIGURE 89. G^* -values for **4.72**

8.5. **4.72.** Autogenerated 2003, 4, 29, 13, 11, 16
Tame game.

9. APPENDIX II: PROVING PRETENDING CORRECT

The results in this section are essentially those of Allemang [A3] (his Lemma 2 and Theorems 1 and 2). We've provided different proofs using our own notation in the hopes of shedding some additional light on them.

9.1. Positions. Let the formal symbol h_i stand for a single heap of size i . A *position* is a sum

$$H = \sum_{i=1}^{\infty} a_i h_i$$

where the a_i are integers ≥ 0 , and all but a finite number of the $a_i = 0$.

We call a_i the *multiplicity* of h_i in H .

The *rank* $r(H)$ is the total number of tokens in H :

$$r(H) = \sum_{i=1}^{\infty} i a_i.$$

9.2. Rank-closed sets. Let C be a set of positions. We call C *rank-closed* if

$$H_1 \in C \Rightarrow H_2 \in C$$

for every position H_2 with $r(H_2) \leq r(H_1)$. If H is a position in a rank-closed set C , then every option of H is also in C .

9.3. Rank-closure operator. Let C be a set of positions. The *rank-closure* \bar{C} is the smallest superset of C that is rank-closed.

9.4. Outcome equivalence. If positions G and H have the same outcome (ie, are both misère N -positions, or are both misère P -positions), we write $G \sim H$.

9.5. Pretending functions. Fix integers $j \geq 1$, $p \geq 1$, and $s \geq 0$.

The *pretending function* $\phi = \phi_{j,p,s}$ acts on positions H as follows:

$$\phi(H) = \begin{cases} H & \text{if } a_j < p + s \\ H - p h_j & \text{otherwise.} \end{cases}$$

In other words, $\phi_{j,p,s}$ reduces the multiplicity of h_j by p —but it does so only if the “input” multiplicity is at least $p + s$.

9.6. Pretending equations. We're interested in pretending functions ϕ that preserve outcome equivalence. Let C be a rank-closed set of positions. If for every $H \in C$ we have

$$H \sim \phi(H),$$

we'll call ϕ a *pretending equation (over C)*. We choose to call such a relation an “equation” because it corresponds to the assertion that the simplification rule

$$\underbrace{h_j + h_j + \dots + h_j}_{p+s \text{ copies}} = \underbrace{h_j + \dots + h_j}_s \text{ copies}$$

preserves outcomes for every position $H \in C$.

9.7. The (p, s) -pretending property. If $\phi_{j,p,s}$ is a pretending equation over C for fixed p and s and every j , then we'll say C has the (p, s) -*pretending property*.

9.8. Depth two lemma. Applying a pretending function ϕ is one way to obtain a simpler position from a more complicated one. Of course, another way to transform a position is to simply make a legal move μ from it. In what follows, we're going to apply the following technical condition that guarantees that μ and ϕ commute:

Lemma 9.1. (*Depth two lemma*) Let $\phi = \phi_{j,p,s}$ be a pretending function, and let H be a position in which the multiplicity a_j of the heap h_j is $\geq 2p + s$. Let μ be an arbitrary move from H . Then $\mu\phi = \phi\mu$.

Proof. By definition, ϕ has no effect on heaps h_i with $i \neq j$. It therefore suffices to consider the effect of μ and ϕ on the a_j heaps of size h_j in H .

We consider two subcases, depending on whether μ is a move from h_j or not:

- (1) Suppose μ is a move from h_j . If μ is applied first to H , then a_j is reduced by one to $a_j - 1$ (some new heaps of various sizes are possibly also added to H as a result of applying the move μ , but that doesn't concern us). Then applying ϕ , a total of

$$a_j - 1 - p \geq 2p + s - 1 - p = p + s - 1 \geq 0$$

copies of h_j will remain in the position.

On the other hand, if we first pretend, then move, ϕ will initially reduce the number of h_j 's by p so that

$$a_j - p \geq 2p + s - p = p + s.$$

remain. Since $p \geq 1$, there's still at least one heap of size h_j left in the position, so now making the move μ , again $p + s - 1$ copies of h_j remain.

- (2) Suppose μ is not a move from h_j . Then applying μ to a position cannot decrease the multiplicity a_j . Let Δ be the change in a_j resulting from applying μ . Then applying μ then ϕ in either order, exactly

$$a_j + \Delta - p \geq 2p + s + \Delta - p = p + s + \Delta \geq p + s > 1$$

heaps h_j will remain. □

Remark 9.2. To see why the condition $a_j \geq 2p + s$ was needed, consider the following example: take $j = 2$, $p = 2$, $s = 0$, and $a_j = 2$ with

$$\begin{aligned} H &= h_2 + h_2 = 2 \cdot h_2 \\ \mu &= 2 \rightarrow 1 \\ \phi &= \phi_{2,2,0} \end{aligned}$$

In this example, we find that $\phi(\mu(H)) = \phi(h_2 + h_1) = h_2 + h_1$. But if ϕ is applied first to H instead, the empty position is immediately obtained. The move μ then can't be made. So ϕ and μ don't commute. This is not a counterexample to the depth two lemma because it has $a_j = 2 < 4 = 2p + s$. The factor of two on p in the expression $2p + s$ is the origin of the term, "depth two."

9.9. Positions to heap size n . Another definition: fix integers $n \geq 1$, $p \geq 1$ and $s \geq 0$. Let C_n be the set of positions

$$C_n = \left\{ \sum_{i=1}^n a_i h_i \mid \text{for } a_i \geq 0 \right\}.$$

C_n is the (infinite) set of positions in which no heap larger than h_n appears.

9.10. Correctly pretending through heap size n . In this section, we prove a theorem relating the correctness of a putative (p, s) -pretending property on the infinite set C_n to its correctness on a particular *finite* subset C . Since a machine (or person!) can ultimately only examine a finite number of outcomes, such a result is an important stepping stone to proving that a particular pretending analysis actually works in practice.

Theorem 9.3. (cf Allemang [A3], Lemma 2 and Theorem 1): *Let $n > 1$, and let $\phi = \phi_{j,p,s}$ be a pretending function. Suppose $C = \bar{D}$ is the rank-closure of the set of positions*

$$D = \left\{ \sum_{i=1}^n a_i h_i \mid \text{each } a_i \text{ satisfies } 0 \leq a_i \leq 2p + s - 1 \right\}.$$

If

C has the (p, s) -pretending property

then

C_n has the (p, s) -pretending property.

Proof. Clearly $D \subseteq C_n$, and supposing the maximum rank attained by an element of D to be R , then

$$C = \{H \mid \text{rank}(H) \leq R\}.$$

C is a finite, rank-closed subset of C_n , and $D \subseteq C$.

The proof is by induction on the rank $r = r(H)$ of a general position $H \in C_n$.

In the base case, if $r \leq R$, then H is already an element of C . There is nothing to prove.

For the induction step, suppose all positions $G \in C_n$ have the (p, s) -pretending property for all ranks less than a fixed constant $k \geq R$. Let H be an arbitrary position at rank $k + 1$.

Because the rank of H is larger than R , we have $H \notin C$, so $H \notin D$. Therefore H has a heap h_j with multiplicity $a_j \geq 2p + s$. The depth-two lemma applies.

Armed with the knowledge that $\mu\phi = \phi\mu$, we consider H . A typical option of H , obtained via a move μ , leads to a position $\mu(H)$, of rank strictly less than $k + 1$. By the induction hypothesis, the outcome of this option can be computed as $\phi(\mu(H))$. Since μ and ϕ commute, this outcome is also obtainable as $\mu(\phi(H))$. But that's just a typical option of $\phi(H)$. So the options of H and $\phi(H)$ are in one-to-one correspondence with the same outcomes. So $H \sim \phi(H)$ for all positions at rank $k + 1$, as desired. \square

9.11. Recurrence of misère values & complete analyses. Our work so far is silent about positions that include heap sizes larger than n tokens. So there is still more work to do.

In 1955, Guy and Smith proved a result about normal play octal games that they stated as follows ([GS], pg 516):

In general, if a game Γ is defined by a finite octal, having P places after the point, and if we can empirically find positive integers p and r_0 such that the equation

$$G(r + p) = G(r)$$

is true for all r in the range $r_0 \leq r < 2r_0 + p + P$, then it is true for all $r \geq r_0$, so that G -function has ultimate period p .

Allemang observed and proved ([A3], Theorem 2, pg. 547–549) that Guy and Smith’s result can be carried over to a similar one for misère play. We need one more definition to state Allemang’s result:

Definition 9.4. Let X be a rank-closed fixed set of games, and suppose that G and H are fixed games. If it’s true that $G + x \sim H + x$ for every $x \in X$, We write the *generalized genus statement*.

$$G \equiv H (X).$$

We can now state Allemang’s result:

Theorem 9.5. For some octal game, let h_n denote the heap of n counters, and let r be the maximum number of counters (r finite) that may be removed from a heap.

If there exist integers m and s such that for i with $m < i \leq 2m + s + r$,

$$h_i \equiv h_{i+s}(C_{2m+2s+r}),$$

then for $i > m$,

$$h_i \equiv h_{i+s}(\text{all positions}).$$

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