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1. Dawson's Chess (**0.137**) and Dawson's Kayles (**0.07**)

In this note, we give a complete analysis of misere *Dawson's Chess* to heap size 30 using quotient semigroup machinery. See [TTW] for background information.

1.1. **Background.** In an unpublished research note [F], Thomas S Ferguson writes:

> Among many imaginative fairy chess problems of T. R. Dawson (1935), Problem 80 asks for the solution of a game that has become to be known as Dawson's Chess.
>
> > *Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, n adjacent files, White to play, at losing game, what is the result?*
>
> It is understood that a capture must be made when possible.
>
> In terms of removing counters from piles, the rules may be described as follows: (1) a pile consisting of a single counter may be removed, (2) two counters may removed from any pile, and (3) three counters may be removed from any pile and if desired that pile may be split into two parts. This game is a member of the class of octal games of Guy and Smith [GS] — specifically **.137** in their notation. Under the rule that the last player to move wins, Guy and Smith show that the game has a remarkable analysis with a Grundy function [nim sequence] eventually periodic of period 34.
>
> However, under the rule proposed by Dawson that the last player to move loses, the game becomes more difficult to analyze. As a partial analysis, Dawson gave some tentative results.
>
> > *For small values of n, up to at least 50, first player loses if n equals 1, 2, 6, 7, or 11, modulus 14. In the case of remainder 4 mod 14, the first player wins whatever move he playes first, e.g. for cases 4, 18, and 32 files.*
>
> Since a straightforward analysis listing winning listing winning positions becomes exceedingly difficult for values of n beyond 20, one wonders how Dawson, so obviously gifted with combinatorial skills, carried out his analysis to $n = 50$.
>
> In 1974 with the aid of a computer at UCLA, it was discovered that Dawson must have made an error in his analysis, since (1) the first player can win with $n = 43$ by moving the central pawn (eliminating the three central files), and (2) for $n = 32$, the first player can make a losing moving by moving the 5th or 11th pawn from the end.
>
> In this note, we show these two facts without the aid of a computer using the analysis of misère games developed by Conway [ONAG]. We then give a complete analysis of Dawson's chess when all piles are of size less than 21…

Dawson himself summarized the situation very well in *Caissa's Wild Roses* ([D], pg 13):

> *This recondite analysis may be commended to any mathematician.*

1.2. **Cousin relations.** Dawson's Kayles (**.07**) and Dawson's Chess (**.137**) are *first cousins* [WWI]—a heap of size n in Dawson's Kayles behaves the same as the heap of size $n-1$ in Dawson's Chess, in both normal and misere play. The game **.4** is a *second cousin* of Dawson's Kayles—a heap of size n in **.4** behaves the same as a heap of size $n-2$ in Dawson's Kayles. Winning Ways (pg 93) points out that that the nim sequence of **.17** is obtained from those for Dawson's Kayles by nim-adding 1 when n is odd.

1.3. **Nim sequences, genera, and canonical forms.** Figure 1 shows genera for **0.07** to heap size 47, and Figure 2 shows its normal play nim sequence, which has period 34 and originally appeared in [GS]. It also appears on pg 89 in [WWI].

This sequence is periodic of length 34 after seven exceptional values

$$G(0) = G(34) = G(14) = 0,$$
$$G(16) = G(17) = G(34) = 2,$$
$$G(31) = 2.$$

	1	2	3	4	5	6	7
0+	0	1	1	2	0	3	1
7+	1	0	3^{1431}	3	2^{0520}	2	4^{146}
14+	0	5^{057}	2^{0520}	2	3^{1431}	3	0^{02}
21+	1^{031}	1^{13}	3^{1431}	0^{31}	2^{0520}	1^{431}	1^{13}
28+	0^{120}	4^{0564}	5^{057}	2^{20}	7^{14875}	4^{57}	0^{02}
35+	1^{031}	1^{13}	2^{1420}	0^{31}	3^{0631}	1^{431}	1^{13}
42+	0^{120}	3^{31}	3^{0531}	2^{20}	2^{1720}		
49	...						

FIGURE 1. $G*$-values for **0.07**

```
n    0 1 2 3 4 5         10                  20                  30
----------------------------------------------------------------------
 0+  0 1 1 2 0 3 1 1 0 3 3 2 2 4 0 5 2 2 3 3 0 1 1 3 0 2 1 1 0 4 5 2 7 4
34+  0 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 2 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4
68+  8 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 9 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4
102+ 8 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 9 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7 4
136+ 8 1 1 ...
```

FIGURE 2. The Remarkable Periodicity of Dawson's Chess

For misère play, [WWI] (pg 418) tabulates 42 terms of the genus sequence and arranges them in rows of 14 values. We've rearranged them into rows of seven values in Figure 1. Conway [ONAG] gives a table (pg 145) of **.4** misère values to heap size 14. Allemang [A1] gives the "blurry" genus sequence of **.07** (in which all nim heap equivalents equal to 4 or greater are replaced by a single # symbol) to heap size 48. The genus sequence of **.17** is obtained from by nim adding 1 to every symbol appearing at an odd heap sizes (see Figure 3).

Figure 4 gives the results of a computer analysis of canonical forms for the first 23 heap sizes of misere Dawson's Kayles, recapitulating information that appears in chapter 13 of [WWI], and which also appears in [F]. Figure 4 uses the following symbols to abbreviate canonical forms:

	1	2	3	4	5	6	7
0+	1	1	0	2	1	3	0
7+	1	1	3^{1431}	2	2^{0520}	3	4^{146}
14+	1	5^{057}	3^{1431}	2	2^{0520}	3	1^{13}
21+	1^{031}	0^{02}	3^{1431}	1^{20}	2^{0520}	0^{520}	1^{13}
28+	...						

FIGURE 3. The $G*$-values for **0.17** are closely related to those of **.07**

```
heap       game   canonical form
h[1]       n[0] = {}
h[2]       n[1] = {n[0]}
h[3]       n[1] = {n[0]}
h[4]       n[2] = {n[0], n[1]}
h[5]       n[0] = {}
h[6]       n[3] = {n[0], n[1], n[2]}
h[7]       n[1] = {n[0]}
h[8]       n[1] = {n[0]}
h[9]       n[0] = {}
h[10]   a4[1,2] = {a[4], n[1], n[2]}
h[11]      n[3] = {n[0], n[1], n[2]}
h[12]     h[12] = {a[5], a4[1, 2], n[0]}
h[13]      n[2] = {n[0], n[1]}
h[14]     h[14] = {h[12], h[17], a[4], n[1], n[3]}
h[15]      n[0] = {}
h[16]     h[16] = {h[14], h[19], g$494, n[0], n[2]}
h[17]     h[17] = {a[5], a4[1, 2], n[0], n[3]}
h[18]      n[2] = {n[0], n[1]}
h[19]     h[19] = {h[12], h[17], a[4], n[1]}
h[20]      n[3] = {n[0], n[1], n[2]}
h[21]    g$2135 = {g$1608, h[14], h[19], a[5], a4[1, 2], n[2]}
h[22]    g$3715 = {g$3172, g$3323, g$3583, h[19], a4[1, 2], n[0], n[3]}
h[23]    g$3962 = {g$2135, h[12], h[16], g$816, g$884, n[3]}
....      ...                     ...
```

FIGURE 4. The increasingly complicated canonical forms of single-heap misere Dawson's Kayles positions at heap sizes 21, 22, and 23 belie its relatively simple underlying misere quotient $\mathcal{Q}_{\mathbf{0.07}}(24)$, which has only twenty-four elements.

(1) h[k] stands for the single heap of size k in misere Dawson's Kayles;
(2) n[k] stands for the misere nim heap of size k;
(3) a[2k] stands for the sum of k copies of the nim heap of size two

$$\mathtt{a}[2\mathtt{k}] = \underbrace{\mathtt{n}[2] + \cdots + \mathtt{n}[2]}_{k \text{ copies}};$$

(4) $\texttt{a[2k+1]} = \texttt{a[2k]} + \texttt{n[1]}$ stands for the sum of k copies of the nim heap of size two with a single nim heap of size 1:

$$\texttt{a[2k+1]} = \underbrace{\texttt{n[2]} + \cdots + \texttt{n[2]}}_{k \text{ copies}} + \texttt{n[1]};$$

(5) $\texttt{a4[1,2]}$ stands for the game with options $\texttt{a[4]}$, $\texttt{n[1]}$, and $\texttt{n[1]}$;

(6) g$xxxx symbols stand for other games whose canonical forms we suppress for the sake of brevity.

The positions in Figure 4 reduce to nim heaps $\texttt{n[k]}$ until heapsize 10, when the game $\texttt{a[4]} = \texttt{n[2]} + \texttt{n[2]}$ first appears as an option of $\texttt{h[10]}$. Amongst the larger heap sizes, the equations

$$\begin{aligned} \texttt{h[17]} &= \texttt{h[10]} + \texttt{n[1]} \\ \texttt{h[19]} &= \texttt{h[12]} + \texttt{n[1]} \\ \text{genus}(\texttt{h[10]}) &= \text{genus}(\texttt{h[19]}) \\ \text{genus}(\texttt{h[12]}) &= \text{genus}(\texttt{h[17]}) \end{aligned}$$

are striking, as well as the appearance of $\texttt{h[17]}$ and $\texttt{h[19]}$ as fully-reduced *options* of $\texttt{h[14]}$ and $\texttt{h[16]}$, respectively.

Because canonical forms for single heaps in Dawson's Kayles get increasingly complicated beginning at heap size 21 (and sums are even worse!), this approach to Dawson's Chess starts to become increasingly unwieldly.

To extend the analysis, we study its quotient semigroup instead.

1.4. **Building up the Dawson quotient $\mathcal{Q}_{\mathbf{0.07}}$.** Let the notation $\mathcal{Q}_{\mathbf{0.07}}(n)$ stand for the misere quotient of Dawson's Chess played with the additional restriction that no heap is larger than size n. We hope to solve the game by finding that these quotient semigroups eventually stabilize.

1.4.1. $\mathcal{Q}_{\mathbf{0.07}}(9)$. We begin with $\mathcal{Q}_{\mathbf{0.07}}(9)$, which is tame with the presentation

$$\mathcal{Q}_{\mathbf{0.07}}(9) = \{x, z \mid x^2 = e,\ z^3 = z\},$$

and the pretending function

	1	2	3	4	5	6	7	8	9
0+	e	x	x	z	e	xz	x	x	e

As usual, we use the symbol e for the identity of the semigroup. The identity corresponds to the endgame. The symbol x corresponds to the misere nim heap of size 1, and z to the misere nim heap of size 2.

$\mathcal{Q}_{\mathbf{0.07}}(9)$ is a six element semigroup with elements

$$\{1, x, z, z^2, xz, xz^2\}.$$

It partitions into P-positions

$$\{x, z^2\}$$

of genera

$$\{1^{031}, 0^{02}\}$$

respectively, and N-positions

$$\{e, z, xz, xz^2\}.$$

with genera

$$\{0^{120}, 2^{20}, 3^{31}, 1^{13}\},$$

respectively.

$\mathcal{Q}_{\mathbf{0.07}}(9)$ is isomorphic to $\mathcal{Q}_{\mathbf{0.333}}$, which we studied in section **??**.

1.4.2. $\mathcal{Q}_{\mathbf{0.07}}(24)$. At heap size 10, Dawson's Kayles becomes wild—the genus of the heap of size 10 is 3^{1431}. Ferguson's note [F] shows how to make assignments of other the nontame-genus heaps of size 20 or less, and it's not hard to extend his information to heap size 24, given our semigroup viewpoint.

The resulting semigroup $\mathcal{Q}_{\mathbf{0.07}}(24)$ has the presentation

$$\mathcal{Q}_{\mathbf{0.07}}(24) = \{x, z, w, v \mid x^2 = e,\ z^4 = z^2,\ w^2 = v^2 = e,\ z^2 = wxz^3\},$$

a semigroup of order 24 (the elements are listed in Figure 5).

e	x	z	w	v	xz
xw	xv	z^2	zw	zv	wv
xz^2	xzw	xzv	xwv	z^3	wz^2
vz^2	zwv	xvz^2	$xzwv$	vz^3	wvz^2

FIGURE 5. The twenty-four elements of $\mathcal{Q}_{\mathbf{0.07}}(24)$.

A Knuth-Bendix rewriting system for $\mathcal{Q}_{\mathbf{0.07}}(24)$ is shown in Figure 6.

$[x^2, e]$	$[w^2, e]$	$[v^2, e]$	$[z^4, z^2]$
$[wz^3, xz^2]$	$[xz^3, wz^2]$	$[xwz^2, z^3]$	

FIGURE 6. A Knuth-Bendix confluent rewriting system for $\mathcal{Q}_{\mathbf{0.07}}(24)$ with 7 reduction rules.

A valid pretending function for all Dawson's Kayles positions with no heap larger than size 24 is

	1	2	3	4	5	6	7	8	9	10
0+	e	x	x	z	e	xz	x	x	e	w
10+	xz	wx	z	v	e	vx	wx	z	w	xz
20+	z^2	x	xz^2	w						

The P-position types are

$$\{x, z^2, wx, vx, vxw\}$$

of genera

$$\{1^{031}, 0^{02}, 2^{0520}, 5^{057}, 6^{064}\},$$

respectively.

1.4.3. $\mathcal{Q}_{\mathbf{0.07}}(29)$. The size of the Dawson quotient doubles from 24 to 48 at heap size twenty five, and this same quotient also works up to heap 29. A presentation is

$$\begin{aligned}\mathcal{Q}_{\mathbf{0.07}}(29) \quad = \quad & \{x, z, w, v, g \mid x^2 = e,\ z^4 = z^2,\ w^2 = v^2 = e,\ g^3 = g^2, \\ & z^2 = wxz^3,\ g^2 = g^2z^2,\ gz = gz^3\}.\end{aligned}$$

The pretending function to heap 29 is

	1	2	3	4	5	6	7	8	9	10
0+	e	x	x	z	e	xz	x	x	e	w
10+	xz	wx	z	v	e	vx	wx	z	w	xz
20+	z^2	x	xz^2	w	g	wx	wz	xz^2	e	

The P position types are

$$\{x, z^2, wx, vx, vxw, g^2, gxz\}$$

of genera

$$\{1^{031}, 0^{02}, 2^{0520}, 5^{057}, 6^{064}, 0^{02}, 3^{02}\},$$

respectively.

1.4.4. $\mathcal{Q}_{\mathbf{0.07}}(31)$. At heap size thirty, the order of the Dawson's Kayles quotient increases to 128. It remains the same size at heap size 31. A valid pretending function to heap size 31 is

	1	2	3	4	5	6	7	8	9	10
0+	e	x	x	z	e	xz	x	x	e	w
10+	xz	$\mathbf{m}$	z	v	e	vx	wx	z	$\mathbf{mx}$	xz
20+	z^2	x	xz^2	w	g	$\mathbf{m}$	$\mathbf{c}$	xz^2	$\mathbf{m}^2$	$\mathbf{h}$
30+	vx									

Three new semigroup generators m, c, and h have been introduced. The positions where they occur are highlighted in bold text. We have

$$\begin{aligned}\mathcal{Q}_{\mathbf{0.07}}(31) \quad = \quad & \{x, z, w, v, g, h, m, c \mid x^2 = e,\ z^4 = z^2, \\ & w^3 = w,\ v^3 = v,\ g^3 = g^2,\ c^4 = c^2,\ h^4 = h^2, \\ & + \textit{84 additional relations involving the 8 generators}\}.\end{aligned}$$

Figure 7 shows a Knuth-Bendix confluent rewriting system for $\mathcal{Q}_{\mathbf{0.07}}(31)$. To determine the outcome class of a general Dawson Kayles position with at most 31 beans per heap, one first applies the pretending function to obtain a monomial in the 8 generators $\{x, z, w, v, g, h, m, c\}$. Next, the resulting word is repeatedly reduced by replacing the left hand term of any applicable reduction rule appearing in Figure 7 by the corresponding right hand side, until no more such reductions are possible. Eventually, one of the 128 canonical $\mathcal{Q}_{\mathbf{0.07}}(31)$ elements shown in Figure 8 is obtained. Amongst these, the P positions correspond to the twenty-seven monomials shown in Figure 9.

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$[c^2, z^2]$	$[z^2w, zc]$	$[xw^2, wm]$	$[w^3, w]$	$[z^2m, xzc]$
$[zm^2, zw^2]$	$[m^3, m]$	$[m^2v, w^2v]$	$[zv^2, z]$	$[v^3, v]$
$[m^2g, w^2g]$	$[mg^2, zg^2]$	$[g^3, g^2]$	$[g^2c, g^2]$	$[g^2h, vg^2]$
$[gh^2, gc]$	$[z^2c, xz^2]$	$[w^2c, c]$	$[m^2c, c]$	$[v^2c, zw]$
$[z^2h, z^2v]$	$[zh^2, xzc]$	$[vh^2, z^2v]$	$[zwc, z^2]$	$[v^2g^2, g^2]$
$[zvgh, zgc]$	$[z^3, xzc]$	$[z^2vh, z^2]$	$[ch^3, ch]$	$[m^2h^2, h^2]$
$[h^4, h^2]$	$[zmc, xz^2]$	$[zch, zvc]$	$[xwm, w^2]$	$[xwc, mc]$
$[xzw, zm]$	$[xwv, mv]$	$[xwg, mg]$	$[zwv, vc]$	$[mvg^2, zvg^2]$
$[xzg^2, zg^2]$	$[z^2vc, zwh]$	$[w^2vc, vc]$	$[wmc, xc]$	$[v^2gc, gc]$
$[zwh^2, xz^2]$	$[zvgc, zgh]$	$[wg^2, zg^2]$	$[xg^2, g^2]$	$[z^2g, gc]$
$[z^2vg, vgc]$	$[xgc, gc]$	$[vch, xz^2]$	$[xz^2v, zwh]$	$[mvh, xzc]$
$[wm^2, w]$	$[w^2m, xw]$	$[wvh, zc]$	$[xzgh, zgh]$	$[xmc, wc]$
$[xzm, zw]$	$[xmv, wv]$	$[xmg, wg]$	$[zmh, z^2v]$	$[zw^2v, wvc]$
$[zwmv, mvc]$	$[zmv, xvc]$	$[xzvg^2, zvg^2]$	$[xvg^2, vg^2]$	$[wvg^2, zvg^2]$
$[gch, vgc]$	$[zwg, gc]$	$[zw^2h, xzvc]$	$[zwmh, zvc]$	$[wmvc, xvc]$
$[xzgc, zgc]$	$[wgc, zgc]$	$[mgc, zgc]$	$[wvgc, zgh]$	$[mvgc, zgh]$
$[xvgc, vgc]$	$[mv^2h, xzvc]$	$[mvgh, zgc]$	$[wv^2h, zvc]$	$[wvgh, zgc]$
$[zw^2g, zgc]$	$[zwmg, zgc]$	$[zmg, gc]$	$[mv^2gh, zgh]$	$[wv^2gh, zgh]$
$[x^2, e]$				

FIGURE 7. A Knuth-Bendix confluent rewriting system for $\mathcal{Q}_{\mathbf{0.07}}(31)$ with 91 reduction rules.

e	x	z	w	m	v	g	c
h	xz	xw	xm	xv	xg	xc	xh
z^2	zw	zm	zv	zg	zc	zh	w^2
wm	wv	wg	wc	wh	m^2	mv	mg
mc	mh	v^2	vg	vc	vh	g^2	gc
gh	ch	h^2	xz^2	xzv	xzg	xzc	xzh
xwh	xm^2	xmh	xv^2	xvg	xvc	xvh	xgh
xch	xh^2	z^2v	zw^2	zwm	zwh	zvg	zvc
zvh	zg^2	zgc	zgh	w^2v	w^2g	w^2h	wmv
wmg	wmh	wv^2	wvg	wvc	wgh	wch	wh^2
m^2h	mv^2	mvg	mvc	mgh	mch	mh^2	v^2g
v^2h	vg^2	vgc	vgh	ch^2	h^3	$xzvg$	$xzvc$
$xzvh$	xwh^2	xm^2h	xmh^2	xv^2g	xv^2h	$xvgh$	xch^2
xh^3	zvg^2	w^2v^2	w^2vg	w^2gh	w^2h^2	wmv^2	$wmvg$
$wmgh$	wmh^2	wv^2g	wch^2	wh^3	mv^2g	mch^2	mh^3
v^2gh	xwh^3	xmh^3	xv^2gh	w^2v^2g	w^2h^3	wmv^2g	wmh^3

FIGURE 8. The 128 elements of $\mathcal{Q}_{\mathbf{0.07}}(31)$.

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x	m	h	xw	xv	z^2	wm	wh	mv
vh	g^2	xzg	xm^2	xmh	xv^2	xh^2	zgc	w^2h
wmv	wgh	m^2h	mv^2	mch	vgh	wmv^2	wh^3	xmh^3

FIGURE 9. The twenty-seven P position types in $\mathcal{Q}_{\mathbf{0.07}}(31)$.

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